# Statistical Mechanics, Gravity, and Euclidean Theory

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**Abstract**

A review of computations of free energy for Gibbs states on stationary but not static gravitational and gauge backgrounds is given. On these backgrounds wave equations for free fields are reduced to eigen-value problems which depend non-linearly on the spectral parameter. We present a method to deal with such problems. In particular, we demonstrate how some results of the spectral theory of second order elliptic operators, such as heat kernel asymptotics, can be extended to a class of non-linear spectral problems. The method is used to trace down the relation between the canonical definition of the free energy based on summation over the modes and the covariant definition given in Euclidean quantum gravity. As an application, high-temperature asymptotics of the free energy and of the thermal part of the stress-energy tensor in the presence of rotation are derived. We also discuss statistical mechanics in the presence of Killing horizons where canonical and Euclidean theories are related in a non-trivial way.

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# 1 Introduction

## 1.1 Basic facts

Quantum field theory at finite temperatures appeared some forty years ago [1]–[3] and now it has many applications ranging from QCD to physics of the early universe, see [4]–[6] for a review. These notes concern one particular aspect of this theory, namely, computation of the effective action and free energy in external gravitational and gauge fields.

A finite-temperature field theory is closely related to Euclidean theory. This fact is now well known and is exploited in numerous applications [4]–[6]. To begin with we recall how to see this relation in a simple way. Consider a model of a real scalar field $\phi$ in Minkowski space-time with the action[1]

$$I_t[\phi] = \frac{1}{2}\int_0^t dt' \int d^3x \left[-(\partial_t\phi)^2 + (\partial_i\phi)^2 + m^2\phi^2\right] \ . \tag{1.1}$$

We study a thermal equilibrium of this field in a finite volume at non-zero temperature $\beta^{-1}$, a Gibbs state. The state is determined by the partition function

$$Z(\beta) = \mathrm{Tr}\ e^{-\beta\hat{H}}, \tag{1.2}$$

where $\hat{H}$ is a normally ordered Hamiltonian of the given model. Normal ordering guarantees that $Z(\beta)$ is well-defined. In general, (1.2) also requires $\hat{H}$ to be an operator bounded from below (which is true for (1.1)). The average $\langle\varphi|e^{-\beta\hat{H}}|\varphi\rangle$ in some quantum state is equivalent to a transition amplitude from this state to itself for the period of time $t = -i\beta$. Such analytical continuation from the real to pure imaginary time is called the Wick rotation. One can define a complete set of states $|\varphi\rangle$ where the field operator $\hat{\phi}$ is diagonal and takes the values $\varphi$. Then matrix elements of the evolution operator can be represented by a path integral [8]

$$\langle\varphi|e^{-it\hat{H}}|\varphi\rangle = \mathcal{N}_t \int D\phi\ e^{-iI_t[\phi]} \ , \tag{1.3}$$

where $\mathcal{N}_t$ is a normalization coefficient and trajectories (in a space of functions) in (1.3) begin and end at $\phi = \varphi$. As a result of the Wick rotation, $I_t[\phi]$ transforms into the functional

$$I^E_\beta[\phi] = iI_{-i\beta}[\phi] = \frac{1}{2}\int_0^\beta d\tau \int d^3x \left[(\partial_\tau\phi)^2 + (\partial_i\phi)^2 + m^2\phi^2\right] \ . \tag{1.4}$$

The latter is the action for the same model but in a flat space-time where the metric is positive definite. One says that such as a space-time has the signature $(++++)$ and calls it Euclidean, as opposed to the physical space-time which has the signature $(-+++)$

[1] We work in the system of units where $\hbar = c = 1$. Our conventions for the curvature and metric coincide with the book [7].

and is called Lorentzian. Now to get (1.2) from (1.3) one has to take the trace over all possible values $\varphi$. Thus, the partition function can be written as

$$\text{Tr}\ e^{-\beta\hat{H}} = \mathcal{N}_\beta \int D\phi\ e^{-I^E_\beta[\phi]}\ . \tag{1.5}$$

By construction, the path integral is taken over all possible closed trajectories going along the Euclidean time $\tau$. This is equivalent to imposing periodicity on $\tau$ with the period $\beta$. This result can be easily extended to non-zero spin fields including fields with Fermi statistics.

The integral in r.h.s. of (1.5) is exponentially suppressed for large values of $\phi$ and this allows its rigorous definition. One can also introduce interaction of the field $\phi$ with an external source. Then the Euclidean path integral can be used as a generating functional for Euclidean Green functions [4],[5]. The Euclidean Green functions correspond to real-time finite-temperature Green functions which obey certain boundary conditions known as Kubo-Martin-Schwinger (KMS) condition [3], see [4]–[6] for more details.

It is instructive to give another derivation of (1.5). Let us introduce the free energy of the system

$$F(\beta) = -\beta^{-1} \ln Z(\beta). \tag{1.6}$$

To calculate it consider a single-particle excitation $\phi_\omega(t,x) = e^{-i\omega t}\phi_\omega(x)$ of the field with a frequency $\omega$. The spectrum of $\omega$ can be found when $\phi_\omega(t,x)$ is substituted in the wave-equation (in Lorentzian space-time). As is easy to see, it yields an eigen-value problem

$$H^2\phi_\omega = \omega^2\phi_\omega, \tag{1.7}$$

$$H^2 = -\partial_i^2 + m^2\ . \tag{1.8}$$

The operator $H = \sqrt{H^2}$ is a relativistic analog of the quantum-mechanical Hamiltonian. In the considered case of a "system in a box" the spectrum is discrete and positive. $F(\beta)$ is well-defined and can be represented in the form [9]

$$F(\beta) = \beta^{-1}\sum_\omega \ln\left(1 - e^{-\beta\omega}\right), \tag{1.9}$$

which is true for free fields.

Let us also define the Euclidean effective action

$$W^E(\beta) = -\ln\int D\phi\ e^{-I^E_\beta[\phi]} = \frac{1}{2}\sum_\Lambda \ln\Lambda\ , \tag{1.10}$$

where $\Lambda$ are eigen-values of the Euclidean operator $L^E = -\partial_\tau^2 - \partial_i^2 + m^2$. The series in (1.10) diverges and as usually has to be regularized according to some prescription. The eigen-vectors of $L_E$ are periodic in $\tau$. They have the form $e^{i\sigma_l\tau}\phi_\omega$ where $\sigma_l = (2\pi l)/\beta$ and $l$ is an integer. Hence, $\Lambda = \sigma_l^2 + w^2$ where $\omega$ are defined by (1.7). The quantities $\sigma_l$ are known in the literature as the Matsubara frequencies [4].

The relation between $F(\beta)$ and $W^E(\beta)$ can be found by using the identity [10],[11]

$$\ln\left(1-e^{-\beta\omega}\right) = -\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta(\nu|\omega,\beta) - \frac{\beta\omega}{2} \ , \tag{1.11}$$

$$\zeta(\nu|\omega,\beta) = \sum_{l=-\infty}^{\infty}\left[\sigma_l^2+\omega^2\right]^{-\nu} \ . \tag{1.12}$$

The $\zeta$-function (1.12) is defined at $\nu = 0$ by analytical continuation from the region $Re\ \nu > 1/2$. By taking into account (1.9) we find that

$$\beta F(\beta) = \lim_{w_0\to\infty}\sum_{w<w_0}\left[-\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta(\nu|\omega,\beta) - \frac{\beta\omega}{2}\right] . \tag{1.13}$$

The first term in r.h.s. of this equation formally coincides with the determinant of the Euclidean operator, see (1.10). The second term in (1.13) is related to the energy of vacuum fluctuations

$$E_0 = \frac{1}{2}\sum_{\omega}\omega \ . \tag{1.14}$$

Hence we conclude that

$$\beta F(\beta) = W^E(\beta) - \beta E_0 \ . \tag{1.15}$$

Although both $W^E$ and $E_0$ diverge their difference is finite and coincides with the free energy. In what follows we give more rigorous definition of $W^E$ and $E_0$. Obviously, (1.15) is in agreement with relation (1.5) derived by the path-integral method where $\ln\mathcal{N}_\beta$ is associated to $\beta E_0$.

In case of constant background fields the Euclidean effective action is closely connected to the notion of effective potential [12] and is a very useful tool in studying phase transitions in a system. Relation (1.15) can be extended to situations when the system is placed in an arbitrary static gravitational field as was first done in pioneering works by Gibbons [13] and Dowker and Kennedy [14] (see also [15] for discussion of (1.15)). This opens the possibility to study a variety of physical phenomena in gravitational fields. In a flat space-time the vacuum energy $E_0$ depends on the boundary conditions, typically on the size of the system. In more general situations $E_0$, like other thermodynamical characteristics, becomes a functional of external background fields. Thus, the difference of $W^E$ and $F(\beta)$ is essential.

We discussed a finite-size system. In many physical situations, however, one has to consider the thermodynamical limit when the size becomes infinitely large. Strictly speaking, in this situation the Gibbs state cannot be realized by a density matrix in a Fock representation. The partition function is not defined because the operator $\exp(-\beta H)$ is not of trace class. As was shown by Haag, Hugenholtz and Winnik [16] the mathematically satisfactory definition of a Gibbs state in this case can be based on the algebraic Gel'fand-Naimark-Segal (GNS) construction. The state is realized as a reducible vector state in GNS representation compatible with KMS condition.

Although the density matrix cannot be introduced for spatially infinite systems the free energy (1.9) and some other quantities like Green functions can still be defined. One can start with a "system in a box" and consider a suitable limit when the box is taken away to infinity. Because of spatial homogeneity free energy of the considered model will be proportional to the volume of the box. Thus, its density per unit volume is well-defined. Analogously one can define the density of $F(\beta)$ when the spatial part of space-time is a hyperbolic manifold of constant negative curvature. This can be done by replacing the summation over the modes in (1.9) with an appropriate integral where the integration measure is determined by the spectral density of the operator $H^2$, see [17] for a review. We will follow a similar method in next sections.

Equation (1.15) will be a starting point of our further discussion. It relates the two definitions of free energy: the first is canonical, $F^C(\beta) = F(\beta)$. It is based on (1.9) and is called "summation over the modes". It is the definition which follows from principles of statistical mechanics and has a clear physical meaning. The second is the definition of free energy in terms of the Euclidean effective action (1.10), i.e., as $F^E(\beta) = \beta^{-1} W^E(\beta)$. The two definitions are related[2] as (see (1.15))

$$F^E(\beta) = F^C(\beta) + E_0. \tag{1.16}$$

Once restricted to static space-times $W^E(\beta)$ and, hence, $F^E(\beta)$ have a number of attractive properties. i) The classical action $I^E_\beta$ has a covariant form in Euclidean space-time. Under a proper choice of integration measure in the functional integral the effective action becomes a covariant functional of the background metric, the property which is especially important in quantum gravity. ii) Euclidean wave operators $L^E$ are positive elliptic operators. There is a number of rigorous mathematical results concerning the properties of these operators [18] which can be used to define $F^E$. iii) Contribution of the vacuum energy $E_0$ is included in $F^E$. This is one of the reasons why $F^E$, as distinct from $F^C$, can be defined as a covariant functional. In principle, like in a classical theory, by varying $F^E$ over the background metric one can define the total finite-temperature stress-energy of a quantum field.

## 1.2 Difficulties

We now discuss situations where the Euclidean definition of free-energy faces difficulties. Consider the same model as before and assume that the field is inside a cylinder of radius $R$. If the cylinder has a finite length we still deal with the field in a box. Let us define the Gibbs state as a thermal equilibrium of the field as measured in the frame of reference which rigidly rotates around the symmetry axes of the cylinder with the angular velocity $\Omega < R^{-1}$. The metric in the rotating frame can be obtained from the Minkowski metric

[2] In some works $F^E$ is also defined without the vacuum energy $E_0$. In this case on static space-times $F^E$ and $F^C$ coincide.

written in polar coordinates $(r, \varphi)$ (with the origin at the center of the cylinder) where the polar angle $\varphi$ should be replaced to $\tilde{\varphi} = \varphi - \Omega t$

$$ds^2 = -(1 - r^2\Omega^2)dt^2 + 2\Omega r^2 dt d\tilde{\varphi} + r^2 d\tilde{\varphi}^2 + dr^2 + dz^2 \ . \tag{1.17}$$

The free energy can be represented in the form (1.9) where summation is taken over frequencies $\tilde{\omega}_l$ of quanta with the angular momentum along the $z$ axis equal to $l$ ($l$ is an integer). If $\omega_l$ is a frequency in the non-rotating frame, then $\tilde{\omega}_l = \omega_l + l\Omega$. This relation is easy to see by rewriting a single-particle solution with momentum $l$ in coordinates (1.17)

$$e^{-i\omega_l t} e^{-il\varphi} \phi_{l,\omega_l}(r, z) = e^{-i\tilde{\omega}_l t} e^{-il\tilde{\varphi}} \phi_{l,\omega_l}(r, z). \tag{1.18}$$

The partition function of the system is defined by (1.2) where $\hat{H}$ is a normally ordered Hamiltonian operator generating time evolution in the rotating frame (1.17). It is not difficult to construct a formal Feynman path integral representation for this partition function and come to equation (1.5). To get the Euclidean action $I^E_\beta$ one has to start from action $I_t$ in Minkowsky space-time, write $I_t$ in coordinates (1.17), go to Euclidean time $\tau = it$ and impose periodicity on $\tau$ with period $\beta$. One finds that

$$I^E_\beta[\phi] = I_{-i\beta}[\phi] = \int_0^\beta d\tau \int r^2 dr d\tilde{\varphi} dz [g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2] \ , \tag{1.19}$$

where $g^{\mu\nu}$ is a contravariant tensor defined as

$$g^{\mu\nu}p_\mu p_\nu = p_\tau^2 + 2i\Omega p_\tau p_{\tilde{\varphi}} + (r^{-2} - \Omega^2)p_{\tilde{\varphi}}^2 + p_r^2 + p_z^2 \ . \tag{1.20}$$

The important difference as compared to the non-rotating ensemble is that the form (1.20) is complex. Thus, the Euclidean classical action in the Feynman path integral (1.20) becomes a complex functional. This results in a difficulty because the corresponding wave operator in the path integral is not a positive-definite elliptic operator[3]. The Euclidean effective action $W^E$ for such complex operators cannot be rigorously defined and the Feynman path integral (1.5) is nothing else but a formal expression.

This problem can be avoided if we simply make an additional step and consider analytical continuation to pure imaginary values of $\Omega = i\breve{\Omega}$. Then the Euclidean action becomes real, the Feynman integral converges, and the Euclidean operator $L_E$ is a positive-definite elliptic operator with the leading symbol

$$\breve{g}^{\mu\nu}p_\mu p_\nu = p_\tau^2 - 2\breve{\Omega}p_\tau p_{\tilde{\varphi}} + (r^{-2} + \breve{\Omega}^2)p_{\tilde{\varphi}}^2 + p_r^2 + p_z^2 \ . \tag{1.21}$$

Now we can interpret our theory as truly Euclidean, i.e. on a real space-time with the signature $(++++)$ and metric

$$ds_E^2 = \breve{g}_{\mu\nu}dx^\mu dx^\nu = (1 + r^2\breve{\Omega}^2)d\tau^2 - 2\breve{\Omega}r^2 d\tau d\tilde{\varphi} + r^2 d\tilde{\varphi}^2 + dr^2 + dz^2 \ . \tag{1.22}$$

[3]Positive-definite elliptic operator is an operator whose leading symbol (1.20) is a positive-definite quadratic form for all $p \neq 0$, see [18].

By going to the new coordinate $\varphi = \tilde{\varphi} - \breve{\Omega}\tau$ one can check that (1.22) is flat. Thus, the Euclidean manifold is a flat $R^4$ space where points $(\tau, \varphi, r, z)$ and $(\tau + \beta, \varphi - \breve{\Omega}\beta, r, z)$ are identified.

This simple flat space example is a good illustration of what happens when one is trying to relate statistical mechanics on an arbitrary stationary space-time to the Euclidean theory. Stationary but not static metrics look as

$$ds^2 = g_{tt}dt^2 + 2g_{ti}dtdx_i + g_{ij}dx^i dx^j \tag{1.23}$$

where components $g_{\mu\nu}$ do not depend on $t$. Such metrics appear in different physical problems. In particular some black hole or cosmological solutions of the Einstein equations (e.g., Kerr, Kerr-Newman or Gödel metrics [19]) have form (1.23). The Wick rotation in this case should be accompanied by an additional analytical continuation of the parameters of the metric to get from (1.23) a real Euclidean metric. This procedure implies complexification of the space-time, the idea which was first suggested and used by Hartle and Hawking [20] and Gibbons and Hawking [21],[22]. More precisely, this procedure can be described as follows [23]. One considers a 4-complex-dimensional manifold $\mathcal{M}_c$ with a complex contravariant tensor field $g_c^{\mu\nu}$ of type (2,0). Lorentzian and Euclidean manifolds are defined as real sections of $\mathcal{M}_c$, i.e., as 2-complex-dimensional submanifolds (slices) on which the restriction of $g_c^{\mu\nu}$ to the cotangent space of the slice is a real form with the signature $(-+++)$ or $(++++)$, respectively. It should be noted that an arbitrary Lorentzian space-time may not admit a complexification which has a real section with Euclidean signature and vice versa. We will exclude these cases from our further discussion.

Although, the described procedure of going to Euclidean theory makes the Feynman integral well-behaved one loses the original relation to statistical mechanics. The problem, thus, is to find out whether there is an analog of equation (1.15) when the space-time is stationary but not static.

There is another problem which is not present in static space-times. In static space-times the spectrum of frequencies $\omega$ is determined by equation (1.7). The latter is obtained from the wave equation for a field under the substitution $\phi_\omega(t, x^i) = e^{-i\omega t}\phi_\omega(x)$. Equation (1.7) is the eigen-value problem for $H^2$ which is a 3-dimensional elliptic operator (see (1.8)). However, on a stationary space-time the same substitution results in equation

$$\left(-g^{tt}(x)\omega^2 + i\omega(-2g^{ti}(x)\partial_i + b(x)) + g^{ij}(x)\partial_i\partial_j + c(x)\right)\phi_\omega(x) = 0, \tag{1.24}$$

where $b(x)$ and $c(x)$ are defined by the metric and parameters of the model. This is an eigen-value problem which depends polinomially on the spectral parameter $\omega$. Equation (1.24) belongs to a class of non-linear spectral problems which appear in different physical situations. A typical example of such a problem is a two-particle scattering where the scattering potential depends on the energy of particles [24]. The difficulty with (1.24) is that its quantum-mechanical interpretation is obscure but, what is worse, this equation is not an eigen-value problem of any operator. Thus, standard results from the theory of elliptic operators cannot be used here. Certainly, it is possible to analyze (1.24) in each particular case and try to solve it explicitly. This is not easy, however[4]. For instance, in case of rotating black holes (1.24) is rather complicated and the fact it allows separation of variables [27] is a big luck. For fields with spins 1/2, 1 and 2 the situation gets more complicated, see [28].

Similar difficulties occur in quantum theory in external static gauge fields. In the presence of electric field the time component $A_t(x)$ of vector potential is not zero and time derivatives in wave equations are covariant derivatives $\partial_t + ieA_t$. To have after the Wick rotation a positive definite elliptic operator one has to go to imaginary potential $A_t = iA_\tau$. This requires complexification of the vector field $A_\mu$. Analogously, like (1.24), the equation for spectrum of frequencies $\omega$ of single-particle states in this situation is not a standard eigen-value problem because it contains terms linear in $\omega$.

Finally, we dwell on another difficulty which appears in case of black holes. A remarkable property of black holes is the Hawking effect [29]: in quantum theory black holes evaporate by emitting particles with a thermal spectrum. The corresponding temperature (the Hawking temperature) is $T_H = \kappa/(2\pi)$ where $\kappa$ is a characteristic of the gravitational field strength near the black hole horizon called the surface gravity (for an introduction to black hole physics see [7],[30]). The Hawking effect implies that a black hole can be in thermal equilibrium with a quantum gas if the temperature of the gas coincides with the Hawking temperature (for a Schwarzschild black hole the corresponding state is known as a Hartle–Hawking vacuum [20]).

This result can be understood on purely geometrical ground. The Schwarzschild metric is a static spherically symmetric metric

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{1.25}$$

where $B(r) = 1 - r_h/r$. The horizon is located at $r = r_h$ and is a 2-sphere of radius $r_h = 2MG$, where $M$ is the mass of a black hole, $G$ is the Newton constant. The Euclidean theory corresponding to a finite-temperature theory around a Schwarzschild black hole is defined on a space obtained from (1.25) by changing $t$ to $-i\tau$ where $\tau$ has to be periodic with a period $\beta$. As is easy to see, near the horizon the Euclidean metric looks as

$$ds_E^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_h^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{1.26}$$

where $\rho \simeq 2\sqrt{r_h(r-r_h)}$ and $\kappa = 1/(2r_h)$ is the surface gravity. The coordinates $\rho, \kappa\tau$ in (1.26) behave as polar coordinates on the plane. If $\beta$ is arbitrary the space with metric

[4]There is a number of publications in mathematical literature where spectrum and eigen functions are studied for one-dimensional non-linear spectral problems similar in the form to (1.24), see, e.g., [25],[26] and references in [26].

$\kappa^2\rho^2 d\tau^2 + d\rho^2$ is a cone. The conical singularity vanishes at $\beta = 2\pi/\kappa$. This corresponds exactly to the Hawking temperature.

If $\beta \neq \beta_H = T_H^{-1}$ the Euclidean space is not smooth. Conical singularities are defects of the geometry where the curvature has delta-function-like behaviour. In the presence of such defects quantum theory acquires new ultraviolet divergencies [31]-[33] in a form of additional divergent terms in the effective action with a support on the Euclidean horizon.

The canonical formulation of statistical-mechanics in the presence of Killing horizons also has non-trivial features. The local temperature measured at a point with coordinate $r$ is given by the Tolman formula, $T(r) = T_H/\sqrt{B(r)}$. The factor $\sqrt{B}$ appears because of the gravitational blue shift of frequencies near the horizon with respect to frequencies at infinity. When $r$ approaches $r_h$ the local temperature becomes infinitely large. So near a black hole one has a thermal bath at very large temperature. In this regime masses of fields play no role and the canonical free energy $F^C$ acquires infrared divergencies.

As a result, in the presence of horizons the Euclidean and the canonical free energies look different and finding the relation between them becomes a problem.

## 1.3 Content

The remaining part of these notes is mainly devoted to two topics where the results are new, i.e. to studying non-linear spectral problems and finding relation between canonical and Euclidean methods for stationary backgrounds. In Section 2, we present an approach which enables one to reduce non-linear spectral problems to standard eigen-value problems of some fiducial elliptic operators. In particular, in section 2.3 we demonstrate that some elements of the spectral geometry can be extended to this class of spectral problems. We introduce an analog of the trace of the heat kernel operator (a pseudo-trace) and show that its asymptotic expansion is determined by local geometrical invariants which generalize the known heat-kernel coefficients. We briefly discuss some physical applications such as high-temperature asymptotics of the free energy and of the stress-energy tensor including the case of non-zero rotation. In Section 3, we use our method to study relation between statistical mechanics and Euclidean theory on arbitrary stationary space-times. Among other advantages, the Euclidean theory allows for an alternative interpretation of the results of Section 2. In Section 4 we discuss possibility to extend the spectral asymptotics to a larger class of non-linear spectral problems. In particular, we demonstrate how it can be done in case of an external electric field. The aim of Section 5 is to consider the features which appear in case of quantum theory in the black hole exterior, or, more exactly, in case of Killing horizons.

It should be emphasized that we are not trying to present mathematically rigorous proofs of our statements but rather use arguments which are closer to physicists. Thus, some of our conclusions can be considered as conjectures. We test, however, how they work in concrete non-trivial examples. Our discussion will be restricted to case of free scalar fields. This enables us to avoid unnecessary complications and concentrate on the main problems. When possible we give references to results concerning higher spin fields.

# 2 Stationary space-times

## 2.1 Killing frame

We begin with some helpful definitions. Consider a field $\phi$ on a domain $\mathcal{M}$ of a $D$-dimensional space-time with a time-like Killing vector[5] field $\xi^\mu$ ($\xi^2 < 0$). $\mathcal{M}$ may be a complete manifold if $\xi^\mu$ is everywhere time-like. In most other cases $\xi^\mu$ is time-like only in some region. We will study solutions of field equations in the frame of reference related to Killing observers whose velocity $u^\mu$ is parallel to $\xi^\mu$

$$u^\mu = B^{-1/2}\xi^\mu \ , \ B = -\xi^2. \tag{2.1}$$

For a Killing observer a solution $\phi_\omega$ carrying the energy $\omega$ is defined as

$$i\mathcal{L}_\xi \phi_\omega = \omega\phi_\omega \tag{2.2}$$

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi^\mu$. The background metric $g_{\mu\nu}$ can be represented as

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu, \tag{2.3}$$

where $h_{\mu\nu}$ is the projector on the directions orthogonal to $u_\mu$. Because the shear and expansion of the family of Killing trajectories vanish identically the trajectories are characterized at each point only by their acceleration $w_\mu$ and the rotation $A_{\mu\nu}$ with respect to a local Lorentz frame [30]

$$w_\mu = u_{\mu;\lambda}u^\lambda \quad , \tag{2.4}$$

$$A_{\mu\nu} = \frac{1}{2}h^\lambda_\mu h^\rho_\nu (u_{\lambda;\rho} - u_{\rho;\lambda}). \tag{2.5}$$

To proceed it is convenient to choose coordinates $x^\mu = (t, x^i)$, $i = 1, D-1$, where $\xi = \partial/\partial t$ and, consequently, $h_{0\mu} = 0$. According with (2.1), (2.3), the interval on $\mathcal{M}$ can be written as

$$ds^2 = -B(dt + a_i dx^i)^2 + dl^2 \ , \tag{2.6}$$

$$-\frac{1}{\sqrt{B}}(u_\mu dx^\mu) = dt + a_i dx^i \ , \tag{2.7}$$

$$a_i = -\frac{u_i}{\sqrt{B}} \ , \tag{2.8}$$

$$dl^2 = h_{\mu\nu}dx^\mu dx^\nu = h_{ij}dx^i dx^j \ . \tag{2.9}$$

[5]By definition the Killing vector is a solution of equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$.

The role of $a_i$ becomes clear under synchronization of clocks by sending light signals: the two events with coordinates $(t, x^i)$ and $(t - a_k dx^k, x^i + dx^i)$ are occuring at the same moment. The metric $dl^2$ serves to measure the proper distance between these points.

In coordinates $(t, x^i)$ the only non-zero components of acceleration (2.4) and rotation (2.5) are

$$w_i = \frac{1}{2}(\ln B)_{,i} \quad , \quad A_{ij} = -\frac{1}{2}\sqrt{B}(a_{i,j} - a_{j,i}). \tag{2.10}$$

In four-dimensional space-time one can define a vector of local angular velocity

$$\Omega_i = \frac{1}{2}\epsilon_{ijk}A^{jk}, \tag{2.11}$$

where $\epsilon_{ijk}$ is a totally antisymmetric tensor. The absolute value of the angular velocity is

$$\Omega = (\Omega_i \Omega^i)^{1/2} = \left(\frac{1}{2}A^{\mu\nu}A_{\mu\nu}\right)^{1/2}. \tag{2.12}$$

The form of the metric in the Killing frame, equations (2.6), (2.7), is preserved under arbitrary change of coordinates $x^i$ provided $h_{ij}$ and $a_i$ transform as a $D-1$ dimensional tensor and a vector. There is also another group of transformations, which preserves (2.6), (2.7), namely, $t = t' + f(x)$, $a_i = a'_i - \partial_i f(x)$, where $f$ is an arbitrary function of $x^i$. Under these transformations $a_i$ changes as an Abelian gauge vector field. By considering single-particle excitations with fixed energy $\omega$

$$\phi_\omega(t, x^i) = e^{-i\omega t}\phi_\omega(x^i) \quad , \tag{2.13}$$

one can realize this group of transformations as a local $U(1)$,

$$\phi_\omega(t, x^i) = \phi'_\omega(t', x^i) = e^{-i\omega t'}\phi'_\omega(x^i) \quad ,$$

$$\phi_\omega(x^i) = e^{i\omega f(x)}\phi'_\omega(x^i) \quad . \tag{2.14}$$

In this picture, $\omega$ coincides with an "elementary charge". To quantize in the Killing frame one needs a full set of modes $\phi_\omega(x^i)$. As follows from the above arguments, the equations which determine $\phi_\omega(x^i)$ have a form of $D-1$ dimensional equations for charged fields in external gauge field $a_i$ on a space with the metric $h_{ij}$. It is important that covariant properties of the theory in $D$ dimensions guarantee diffeo- and gauge-covariant form of the $D-1$ dimensional problem. Such a reduction from $D$ to $D-1$ is analogous to the Kaluza–Klein procedure which yields the Einstein–Maxwell theory from higher dimensional gravity. The difference between the two reductions is that in the standard Kaluza–Klein approach the "extra" dimensions are compact and the charges are quantized.

Let $\mathcal{B}$ be a $D-1$ dimensional space with metric $dl^2$, see (2.9). Consider a point $p$ on $\mathcal{B}$ with coordinates $x^i$ and a vector $V_i$ from the tangent space at $p$. On $\mathcal{M}$, $p$ corresponds to a trajectory of a Killing observer with the same coordinates $x^i$. At any point of the trajectory one can define a vector $V_\mu$ orthogonal to $u^\mu$ such as $V_i = h_i^\mu V_\mu$. Suppose that connection $\tilde{\nabla}_i$ on $\mathcal{B}$ is determined by $h_{ij}$. Then the covariant derivative with respect to this connection can be written as [19]

$$\tilde{\nabla}_j V_i = h_i^\lambda h_j^\rho V_{\lambda;\rho} \quad , \tag{2.15}$$

where $V_{\mu;\nu}$ is the covariant derivative on $\mathcal{M}$ with respect to the connection defined by $g_{\mu\nu}$. (One can easily check that $\tilde{\nabla}_k h_{ij} = 0$.) Relation (2.15) can be generalized to an arbitrary field on $\mathcal{M}$. For instance, for a scalar field

$$h_j^\mu \partial_\mu \phi = (\partial_j - a_j \partial_t)\phi \equiv D_i \phi \quad , \tag{2.16}$$

for a vector orthogonal to $u^\mu$

$$h_i^\lambda h_j^\rho V_{\lambda;\rho} = (\tilde{\nabla}_j - a_j \partial_t) V_i \equiv D_j V_i \quad , \tag{2.17}$$

where $V_i = h_i^\mu V_\mu$. The time derivative in (2.16), (2.17) appears in general because fields on $\mathcal{M}$ change along the Killing trajectory. If $\phi$ and $V_\mu$ are solutions with certain frequency, see (2.13), then $D_i$ become covariant derivatives on $\mathcal{B}$ in external gauge field $a_i$. This demonstrates explicitly diffeo- and gauge-covariance of the theory which are left after the reduction.

## 2.2 Kaluza-Klein method

We now present an approach developed by Fursaev in [34] [6]. Consider a real scalar field $\phi$ which satisfies the equation

$$(-\nabla^\mu \nabla_\mu + V)\phi = 0, \tag{2.18}$$

where $V$ is a potential. Suppose that $\mathcal{M}$ is a globally hyperbolic space-time[7] and $\Sigma$ is a Cauchy hypersurface in $\mathcal{M}$. One can define the Klein–Gordon inner product on $\Sigma$ (see [35])

$$< \phi_1, \phi_2 > \equiv \int_\Sigma d\Sigma^\mu j_\mu(\phi_1, \phi_2) \ , \tag{2.19}$$

$$j_\mu(\phi_1, \phi_2) = -i(\phi_1^* \partial_\mu \phi_2 - \partial_\mu \phi_1^* \phi_2). \tag{2.20}$$

If $\phi_1$ and $\phi_2$ are solutions to (2.18) the current $j_\mu$ is divergence free, $\nabla^\mu j_\mu = 0$. This property guarantees that (2.19) is independent of the choice of $\Sigma$.

Consider first systems in a finite volume (space $\mathcal{B}$ is compact). Then the problem is formulated as follows: one has to find a complete set of solutions $\phi_\omega(t, x^i)$ to (2.18) with certain frequencies $\omega$ which are ortho-normalized with respect to the product (2.19).

[6] We discuss scalar fields only. Ref. [34] also includes the analysis of the spinor fields but has a mistake in the definition of a spinor Hamiltonian. Thus its results for spinor fields require a revision.

[7] A globally hyperbolic space-time is a one which admits a Cauchy surface. A Cauchy surface $\Sigma$ is a space-like hypersurface such that every non-space-like curve intersects $\Sigma$ exactly once. For more details see [19].

Frequencies $\omega$ are energies of single-particle excitations. The spectrum of $\omega$ can be used to define the canonical free energy $F^C$ in the Gibbs state by (1.9).

Let us demonstrate how finding the spectrum can be reduced to a standard eigen-value problem for an elliptic operator. By using the results of the preceding section the wave operator in the Killing frame (2.3) can be represented as

$$\nabla^\mu\nabla_\mu=\tilde{g}^{\mu\nu}D_\mu D_\nu, \tag{2.21}$$

$$D_\mu=\tilde{\nabla}_\mu-a_\mu\partial_t, \tag{2.22}$$

where $a_\mu dx^\mu=a_i dx^i$. The connections $\tilde{\nabla}_\mu$ are determined on some space $\tilde{\mathcal{M}}$ with metric

$$d\tilde{s}^2=\tilde{g}_{\mu\nu}dx^\mu dx^\nu=-Bdt^2+dl^2. \tag{2.23}$$

Relation between $\tilde{\mathcal{M}}$ and $\mathcal{M}$ becomes transparent when comparing (2.23) with (2.6). We will call $\tilde{\mathcal{M}}$ and $a_\mu$ the fiducial space-time and the fiducial gauge potential, respectively. Let us consider now a scalar field $\phi^{(\lambda)}$ on $\tilde{\mathcal{M}}$ which obeys the equation

$$(-\tilde{g}^{\mu\nu}(\tilde{\nabla}_\mu+i\lambda a_\mu)(\tilde{\nabla}_\nu+i\lambda a_\nu)+V)\phi^{(\lambda)}=0, \tag{2.24}$$

where $\lambda$ is a real parameter. Because (2.24) does not contain terms linear in the time derivatives, it takes the following form

$$H^2(\lambda)\phi^{(\lambda)}_\omega(x^i)=\omega^2(\lambda)\phi^{(\lambda)}_\omega(x^i) \tag{2.25}$$

on functions $\phi^{(\lambda)}_\omega(t,x^i)=e^{-i\omega t}\phi^{(\lambda)}_\omega(x^i)$. The operator $H(\lambda)$ has the meaning of a relativistic Hamiltonian for the field $\phi^{(\lambda)}_\omega(x^i)$ on $\mathcal{B}$. It takes the simplest form after the transformation (which does not change the spectrum)

$$\bar{H}(\lambda)=e^{-\frac{D-2}{2}\sigma}H(\lambda)e^{\frac{D-2}{2}\sigma}, \tag{2.26}$$

$$\bar{H}^2(\lambda)=-\bar{h}^{ij}(\bar{\nabla}_i+i\lambda a_i)(\bar{\nabla}_j+i\lambda a_j)+\bar{V}, \tag{2.27}$$

$$e^{-2\sigma}=-\xi^2=B. \tag{2.28}$$

Connections $\bar{\nabla}_i$ correspond to fields on a $D-1$ dimensional space $\bar{\mathcal{B}}$ conformally related to $\mathcal{B}$

$$d\bar{l}^2=\bar{h}_{ij}dx^i dx^j=e^{2\sigma}dl^2. \tag{2.29}$$

The "potential term" in (2.27) is

$$\bar{V}=B\left[V+\frac{D-2}{2}(\nabla^\mu w_\mu-\frac{D-2}{2}w^\mu w_\mu)\right], \tag{2.30}$$

where $w_\mu$ is the acceleration (2.4). One can arrive at (2.27) by doing a conformal transformation in equations (2.18), (2.24). Under this transformation the physical metric $g_{\mu\nu}$ changes to $\bar{g}_{\mu\nu}=g_{\mu\nu}/B$ and the Killing vector on the rescaled space has the unit norm, $\xi^2=-1$.

By taking into account (2.27) we conclude that (2.25) is the standard eigen-value problem in a finite volume for a positive-definite elliptic operator. The spectrum of $H^2(\lambda)$ is discrete and bounded from below [18]. For simplicity we assume that the spectrum is strictly positive.

Because the fiducial spectrum depends on $\lambda$ we can find the physical spectrum from the constraint

$$\chi(\lambda,\omega)=0,\ \ \lambda>0, \tag{2.31}$$

$$\chi(\lambda,\omega)=-\omega^2(\lambda)+\lambda^2. \tag{2.32}$$

If $\lambda$ obeys (2.31) we denote the corresponding solution to (2.24) as $\phi_\omega^{(\omega)}(t,x^i)=e^{-i\omega t}\phi_\omega^{(\omega)}(x^i)$. As follows from (2.21)–(2.25), (2.31) this field is also a solution to physical equation (2.18), i.e. one can write

$$\phi_\omega(t,x^i)=C_\omega\phi_\omega^{(\omega)}(t,x^i), \tag{2.33}$$

where $C_\omega$ is a normalization coefficient. Therefore, the relativistic eigen-value problem can be reduced to the standard eigen-value problem for a one-parameter family of elliptic operators $H^2(\lambda)$ and to solving (2.31). We call this method of finding solutions to (2.18) the Kaluza–Klein (KK) method.

To fix $C_\omega$ we have to analyze the inner products. The fiducial fields obey equations (2.24) which dictate a different form of the corresponding vector current and the product

$$\tilde{j}_\mu(\phi_1,\phi_2)=-i(\phi_1^*(\partial_\mu+i\lambda a_\mu)\phi_2-(\partial_\mu-i\lambda a_\mu)\phi_1^*\phi_2), \tag{2.34}$$

$$(\phi_1,\phi_2)\equiv\int_{\tilde{\Sigma}}d\tilde{\Sigma}^\mu\tilde{j}_\mu(\phi_1,\phi_2). \tag{2.35}$$

The connection between (2.19) and (2.35) can be established as follows. (Without loss of generality we put $B=-\xi^2=1$ because one can always reduce the problem to this form by conformal rescaling.) Choose $\Sigma$ and $\tilde{\Sigma}$ as constant time hyper-surfaces[8]. Then

$$<\phi_\omega,\phi_\sigma>=\int_\Sigma\sqrt{h}d^{D-1}x\left[(\omega+\sigma)\phi_\omega^*\phi_\sigma+i\phi_\omega^*a^i(\nabla_i+i\sigma a_i)\phi_\sigma-i(\nabla_i-i\omega a_i)\phi_\omega^*a^i\phi_\sigma\right], \tag{2.36}$$

$$(\phi_\omega^{(\lambda)},\phi_\sigma^{(\lambda)})=(\omega+\sigma)\int_{\tilde{\Sigma}}\sqrt{h}d^{D-1}x(\phi_\omega^{(\lambda)})^*\phi_\sigma^{(\lambda)}, \tag{2.37}$$

where $h=\det h_{ij}$, and $h_{ij}$ is the metric on $\mathcal{B}$. Indices $i,j$ are raised with the help of $h^{ij}$. Note that up to a multiplier (2.37) coincides with standard product on space of square integrable functions $L^2$. Equations (2.36), (2.37) can be used to write

$$\begin{aligned}<\phi_\omega,\phi_\sigma>&=(\phi_\omega,\phi_\sigma)\\&+\int_\Sigma\sqrt{h}d^{D-1}\left(i\phi_\omega^*a^i(\nabla_i+i\sigma a_i)\phi_\sigma-ia^i(\nabla_i-i\omega a_i)\phi_\omega^*\phi_\sigma\right).\end{aligned} \tag{2.38}$$

When $\omega\neq\sigma$ one can use the identity

$$H^2(\omega)-H^2(\sigma)=(\omega-\sigma)(-2ia^i\nabla_i-i\nabla_i a^i+(\omega+\sigma)a^ia_i). \tag{2.39}$$

[8]For fiducial space-time $\tilde{\mathcal{M}}$ metric induced on $\tilde{\Sigma}$ coincides with metric on $\mathcal{B}$.

to rewrite (2.38) as

$$< \phi_\omega, \phi_\sigma >= (\phi_\omega, \phi_\sigma)$$
$$+\frac{1}{2(\sigma^2-\omega^2)}\left[(\phi_\sigma,(H^2(\omega)-H^2(\sigma))\phi_\omega)^* + (\phi_\omega,(H^2(\omega)-H^2(\sigma))\phi_\sigma)\right]. \qquad (2.40)$$

The r.h.s. of (2.40) vanishes if $H^2(\lambda)$ is Hermitean, which we assume to be the case. (2.40) demonstrates that physical modes $\phi_\omega$, $\phi_\sigma$ obtained from fiducial modes under restriction (2.31) are orthogonal with respect to (2.36) when $\omega \neq \sigma$.

Consider now the modes with equal frequencies, $\phi_\omega$ and $\tilde{\phi}_\omega$. (2.38) can be written as

$$< \tilde{\phi}_\omega, \phi_\omega >= (\tilde{\phi}_\omega, \phi_\omega) + i\int_\Sigma \sqrt{h} d^{D-1}x \tilde{\phi}^*_\omega \left[2a^i(\partial_i + i\omega a_i) + \nabla^i a_i\right]\phi_\omega. \qquad (2.41)$$

By differentiating both sides of (2.25) with respect to $\lambda$ one finds

$$\partial_\lambda H^2(\lambda)\phi^{(\lambda)}_\omega + H^2(\lambda)\partial_\lambda\phi^{(\lambda)}_\omega = \partial_\lambda\omega^2(\lambda)\phi^{(\lambda)}_\omega + \omega^2(\lambda)\partial_\lambda\phi^{(\lambda)}_\omega, \qquad (2.42)$$

$$\partial_\lambda H^2(\lambda) = -2ia^i(\nabla_i + i\lambda a_i) - i\nabla^i a_i. \qquad (2.43)$$

Relation (2.41) can be rewritten by using (2.42), (2.43) and definition (2.31) as

$$< \tilde{\phi}_\omega, \phi_\omega >= \frac{\chi'(\omega,\omega)}{2\omega}(\tilde{\phi}_\omega, \phi_\omega), \qquad (2.44)$$

where $\chi'(\omega,\omega) = \partial_\lambda\chi(\lambda,\omega)$ at $\chi(\lambda,\omega) = 0$. Because the form $(\tilde{\phi}_\omega, \phi_\omega)$ is positive-definite it follows from (2.44) that the form determined by the Klein-Gordon product is positive-definite if

$$\chi'(\omega,\omega) > 0 \ . \qquad (2.45)$$

If $\chi'(\omega,\omega) < 0$ for some $\omega$ the corresponding state has negative Klein-Gordon norm and should be excluded. We will assume that condition (2.45) is satisfied for any solution of (2.31). We will see later that (2.45) also plays a role in the Euclidean formulation of the theory.

If (2.45) is satisfied then the orthogonality of $\tilde{\phi}_\omega$, $\phi_\omega$ with respect to the standard product implies their orthogonality with respect to (2.44). Finally, if $\phi^{(\omega)}_\omega$ have a unit norm, then according to (2.44), normalization of $\phi_\omega$ in (2.33) requires $|C_\omega|^{-2} = \chi'(\omega,\omega)/(2\omega)$.

To summarize: we have proved that functions (2.33) are a set of modes orthonormalized with respect to the Klein–Gordon inner product provided the fiducial modes are orthonormalized with respect to (2.35). This fact also means that there is no contradiction between orthonormalization procedures for physical and fiducial modes. Given this result, one can go further and study other propertiers of the set of physical modes including its completeness [9].

[9]For one-dimensional non-linear spectral problems (for Sturm-Liouville operator) the completeness of eigen-functions is discussed in [26].

## 2.3 Spectral coefficients

We discuss now properties of the spectrum of single-particle states. For the class of models considered in section 2.2 one can define the function

$$K(t) = \sum_{\omega} e^{-t\omega^2}, \tag{2.46}$$

where $t > 0$. If the space-time is static (2.46) coincides with the trace of the heat kernel of the operator $H^2$,

$$K(t) = \mathrm{Tr}\ e^{-tH^2}. \tag{2.47}$$

In general, however, $K(t)$ is not a trace of any operator like (2.47). Let us call it a pseudo-trace. For an elliptic operator $H^2$ the asymptotic of (2.47) at small $t$ has the known form. To avoid unnecessary complications we assume that the system is defined on a compact space without boundaries. Then at small $t$

$$\mathrm{Tr}\ e^{-tH^2} \sim \frac{1}{(4\pi t)^{(D-1)/2}} \sum_{n=0}^{\infty} a_n t^n. \tag{2.48}$$

The coefficients $a_n$ are called Hadamard–Minackshisundaram–DeWitt–Seeley coefficients, or sometimes the heat coefficients. They are local invariant functionals polynomial in curvatures of the background metric.

The expansion (2.48) plays an important role in mathematical [18] and physical applications. It is natural to ask whether there is an analog of this expansion for pseudo-trace $K(t)$ in general case.

To answer this question let us note that $K(t)$ diverges at small $t$ as a result of summation over arbitrary large $\omega^2$. When $t$ is finite there is the exponential cutoff of the series at $\omega^2 \sim 1/t$. Therefore, asymptotic properties of $K(t)$ are related to the behaviour of the spectrum at large $\omega^2$, the same property which determines the asymptotics of the heat kernel in (2.48). For large $\omega$ the spectrum becomes sufficiently dense and can be considered continuous. We will use this fact for our purpose. Define $K(t)$ for operators with continuous spectrum as

$$K(t) = \int_{\mu}^{\infty} \Phi(\omega) d\omega e^{-t\omega^2}. \tag{2.49}$$

Here $\mu$ is the mass gap of the spectrum. To avoid complications we also assume that $\mu > 0$ and there are no isolated discrete levels[10]. The quantity $\Phi(\omega)$ is the spectral density or the total spectral measure defined on a compact domain $\Sigma_r$ of a Cauchy hypersurface $\Sigma$ as

$$\Phi(\omega) = \int_{\Sigma_r} d\Sigma^{\mu} \sum_{l} j_{\mu}(\phi_{\omega,l}, \phi_{\omega,l}), \tag{2.50}$$

[10]These assumptions seem to be not essential in studying the limit of large $\omega$.

where $j_\mu$ is defined in (2.20). We assume that $\phi_{\omega,l}$ is a complete set of single-particle solutions to (2.18). Index $l$ enumerates all modes with the same frequencies. In problems with a continuous spectrum $\Sigma$ has an infinite volume and normalization of modes is to be understood in terms of distributions, i.e., as

$$< \phi_{\omega,k}, \phi_{\sigma,l} >= \delta_{lk}\delta(\omega - \sigma), \tag{2.51}$$

where $\delta(x)$ is the delta-function. The integral on r.h.s. of (2.50) is divergent. This is the divergence of the large volume or infrared divergence. To avoid the divergence one has to work with a regularized density obtained by restricting the integration in (2.50) to some compact domain $\Sigma_r$ in $\Sigma$ and then expanding $\Sigma_r$ to $\Sigma$ [11].

The important advantage of the KK-method is that the spectral density $\Phi(\omega)$ can be related to the spectral density of the corresponding fiducial problem (2.25). Suppose that $\phi^{(\lambda)}_{\omega,k}$ are a complete set of ortho-normalized solutions of (2.25)

$$(\phi^{(\lambda)}_{\omega,k}, \phi^{(\lambda)}_{\sigma,l}) = \delta_{lk}\delta(\omega - \sigma). \tag{2.52}$$

If the modes are related by

$$\phi_\omega(t, x^i) = \phi^{(\omega)}_\omega(t, x^i), \tag{2.53}$$

one can show [34] by using (2.36) and (2.37) that the normalization (2.51) follows from (2.52).

The (regularized) spectral density of fiducial modes is

$$\Phi(\omega; \lambda) = \int_{\tilde{\Sigma}_r} d\Sigma^\mu \sum_l \tilde{j}_\mu(\phi^{(\lambda)}_{\omega,l}, \phi^{(\lambda)}_{\omega,l}), \tag{2.54}$$

where $\tilde{j}_\mu$ is defined in (2.34) and $\tilde{\Sigma}_r$ is a compact domain in $\tilde{\Sigma}$. The definition of $\tilde{\Sigma}_r$ corresponds to the definition of the domain $\Sigma_r$ in $\Sigma$ (we choose again $\Sigma$ and $\tilde{\Sigma}$ as constant time hypersurfaces). Let us introduce also an auxiliary quantity

$$\Psi(\omega; \lambda) = \Phi(\omega; \lambda) - \frac{1}{4\lambda} \sum_k \left[ (\phi^{(\lambda)}_{\omega,k}, \partial_\lambda H^2(\lambda)\phi^{(\lambda)}_{\omega,k}) + (\phi^{(\lambda)}_{\omega,k}, \partial_\lambda H^2(\lambda)\phi^{(\lambda)}_{\omega,k})^* \right], \tag{2.55}$$

where $\partial_\lambda H^2(\lambda)$ is given in (2.43). As follows from (2.38),

$$\Phi(\omega) = \Psi(\omega; \omega). \tag{2.56}$$

It should be noted that $\Phi(\omega)$ does not coincide with $\Phi(\omega; \omega)$, as one could naively expect. The distinction of these two quantities is in different forms of the inner products. Consider now the spectral representation for the heat kernel of $H^2(\lambda)$

$$\mathrm{Tr}\, e^{-tH^2(\lambda)} = \int_\mu^\infty \Phi(\omega; \lambda) e^{-t\omega^2} d\omega, \tag{2.57}$$

[11] We will see that this procedure enables one to study local characteristics of the system such as the density of the free energy per unit volume or a stress-energy tensor.

where integration in the trace is restricted by $\tilde{\Sigma}_r$. We assume that the mass gap of $H^2(\lambda)$ is positive and does not depend on $\lambda$ (which is true in a number of physical problems). Similarly we can define the integral

$$\int_\mu^\infty \Psi(\omega;\lambda)e^{-t\omega^2}d\omega = \mathrm{Tr}\left[\left(1-\frac{1}{2\lambda}\partial_\lambda H^2(\lambda)\right)e^{-tH^2(\lambda)}\right], \tag{2.58}$$

where the r.h.s. is the consequence of (2.55). Because the trace does not depend on the choice of the basis and, hence, on $\lambda$ one can write (2.58) as

$$\int_\mu^\infty \Psi(\omega;\lambda)e^{-t\omega^2}d\omega = \left(1+\frac{1}{2\lambda t}\partial_\lambda\right)\mathrm{Tr}\ e^{-tH^2(\lambda)}. \tag{2.59}$$

The latter relation is equivalent to

$$\Psi(\omega;\lambda) = \Phi(\omega;\lambda) + \frac{\omega}{\lambda}\int_\mu^\omega \partial_\lambda\Phi(\sigma;\lambda)d\sigma \quad . \tag{2.60}$$

Formula (2.60) is our key relation which together with (2.56) enables one to compute the physical spectral density $\Phi(\omega)$ by using powerful heat kernel techniques. It is remarkable that (2.60) reappears in the Euclidean theory (see section 3.3).

Consider now a short $t$ asymptotics of (2.57)

$$\mathrm{Tr} e^{-tH^2(\lambda)} \sim \frac{1}{(4\pi t)^{(D-1)/2}}\sum_{n=0}^\infty a_n(\lambda)t^n. \tag{2.61}$$

The coefficients $a_n(\lambda)$ are heat kernel coefficients for the operator (2.27). They are local functionals on $\tilde{\Sigma}_r$ and are even in $\lambda$ (the fiducial theory is $U(1)$ invariant and the heat coefficients are even functions of charges). The gauge invariance also guarantees that the coefficients are polynomials in powers of the Maxwell stress tensor and its derivatives. In our case the role of the gauge field is played by the vector $a_i$, and the corresponding Maxwell tensor $F_{ij} = a_{j,i} - a_{j,i}$ is related to the rotation. In general,

$$a_n(\lambda) = \sum_{m=0}^{[n/2]} \lambda^{2m} a_{2m,n} \quad , \tag{2.62}$$

where $a_{2m,n}$ do not depend on $\lambda$. The highest power of $\lambda$ in (2.62) can be determined by dimensional analysis. Coefficients $a_0$ and $a_1$ in (2.61) do not depend on $\lambda$.

The spectral density at high frequencies can be found from (2.61) by using (2.59). One can neglect for simplicity the mass gap and use inverse Laplace transform in (2.59). It should be noted, however, that for operators with zero gap one has to take into account the presence of infrared singularities which come out in (2.57) at small $\omega$. One of the possibilities to avoid this problem is to use dimensional regularization and formally consider $D$ as a complex parameter.

It is instructive first to obtain the asymptotics for the fiducial spectral density at large $\omega$

$$\Phi(\omega;\lambda) \sim \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}}\sum_{n=0}^\infty \frac{a_n(\lambda)}{\Gamma\left(\frac{D-1}{2}-n\right)}\omega^{-2n}. \tag{2.63}$$

One can easily verify that for complex $D$ substitution of (2.63) in (2.57) yields (2.61). Another method how to define (2.63) for integer $D$ is discussed in [36]. For $\Psi(\omega;\lambda)$ relation (2.59) results in expansion

$$\Psi(\omega;\lambda) \sim \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}} \sum_{n=0}^{\infty} \frac{\tilde{a}_n(\lambda)}{\Gamma\left(\frac{D-1}{2}-n\right)}\omega^{-2n}, \tag{2.64}$$

$$\tilde{a}_n(\lambda) = a_n(\lambda) + \frac{1}{2\lambda}\partial_\lambda a_{n+1}(\lambda). \tag{2.65}$$

Finally, by taking into account (2.56), (2.62), (2.64), (2.65) one finds

$$\Phi(\omega) \sim \frac{2\omega^{D-2}}{(4\pi)^{(D-1)/2}} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma\left(\frac{D-1}{2}-n\right)}\omega^{-2n}, \tag{2.66}$$

$$c_n = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma\left(m-\frac{D-1}{2}\right)}{\Gamma\left(n-\frac{D-1}{2}\right)} a_{2(m-n),m}, \tag{2.67}$$

We call $c_n$ the spectral coefficients. It is remarkable that they are local functionals expressed in terms of heat-kernel coefficients of elliptic operators.

Now some comments are in order. Formula (2.67) is valid in any dimension $D$. As for expansions (2.63) and (2.66), they are finite in even dimension $D = 2k$. For $D = 2k+1$ terms in (2.66) with $n \geq k$ are formally zero. This indicates that one should be more careful with infrared singularities in this case. The proper way of dealing with the infrared problem is to keep $D$ complex in (2.66) till the last stage of computations. Then for any $D$ some relevant quantities determined with the help of $\Phi(\omega)$ are finite except, possibly, a number of standard poles (see section 2.5).

We return now to the question about asymptotics of the pseudo-trace $K(t)$. To find its behavior at small $t$ it is enough to substitute (2.66) in (2.49). By neglecting the gap we get the expansion

$$K(t) \sim \frac{1}{(4\pi t)^{(D-1)/2}} \sum_{n=0}^{\infty} c_n t^n. \tag{2.68}$$

As was expected, (2.68) holds both for odd and even $D$ because infrared singularities play no role. We conjecture that the pseudo-trace expansion (2.68) has to be equally valid for continuous and discrete spectra.

The coefficients in leading terms in (2.68) can be immediately computed by using (2.67). For instance, $c_0 = a_0$ and

$$c_1 = a_{0,1} + \frac{D-3}{2} a_{2,2}. \tag{2.69}$$

Term $a_{2,2}$ is determined by the gauge part of $a_2(\lambda)$, see (2.62),

$$a_{2,2} = -\frac{1}{12}\int_{\Sigma_r} h^{1/2} d^{D-1}x F^{ij} F_{ij}, \tag{2.70}$$

where $F_{ij} = a_{j,i} - a_{i,j}$ is a "Maxwell tensor" of the fiducial gauge field. Term $a_{0,1}$ coincides with $a_1$ in expansion (2.61) for operator (2.27) with $\lambda = 0$.

## 2.4 Examples

There are simple examples where our results can be checked. One example is motivated by studying a quantum theory in the Einstein universe $R^1 \times S^{D-1}$. Consider a Gibbs state in this space-time defined as a thermal equilibrium in the frame of reference which rigidly rotates with angular velocity $\Omega_0$ (with restriction $\Omega_0 < 1/\rho$ where $\rho$ is the radius of $S^{D-1}$). The spectrum can be easily found for conformal scalar fields

$$(-\nabla^2 + \xi_D R)\phi = 0, \tag{2.71}$$

where $\xi_D = (D-2)/(4(D-1))$ and $R = D(D-1)/\rho^2$ is the scalar curvature. Suppose for simplicity that $D = 3$ and $\rho = 1$. The metric is

$$ds^2 = -dt^2 + \sin^2\theta d\varphi^2 + d\theta^2. \tag{2.72}$$

In the rotating frame this metric can be written as

$$ds^2 = -B(dt + a_\varphi d\tilde{\varphi})^2 + \frac{\sin^2\theta}{B} d\tilde{\varphi}^2 + d\theta^2, \tag{2.73}$$

where $\tilde{\varphi} = \varphi - \Omega_0 t$ and

$$B = 1 - \Omega_0^2 \sin^2\theta,\ \ a_\varphi = \Omega_0 \sin^2\theta B^{-1}. \tag{2.74}$$

The spectrum of frequencies in a non-rotating frame can be easily found, $\omega_n = n + 1/2$, where $n = 0, 1, ..$ is the angular momentum of a particle on $S^2$. The frequencies in the rotating frame are $\tilde{\omega}_{nl} = \omega_n + \Omega_0 l$ (see (1.18)), where $|l| \leq n$. Thus, the pseudo-trace $K(t)$ is defined as

$$K(t) = \sum_{n=0}^{\infty} \sum_{l=-n}^{n} e^{-\tilde{\omega}_{nl}^2 t}. \tag{2.75}$$

The first spectral coefficients in the asymptotic expansion (2.68) for (2.75) can be found by a direct computation[12]

$$c_0 = \frac{4\pi}{1-\Omega_0^2}\ ,\ \ c_1 = \frac{\pi}{3}\left[2 - \frac{1}{1-\Omega_0^2}\right]. \tag{2.76}$$

We show now that results (2.76) are reproduced by formula (2.67) and $c_0$, $c_1$ can be represented as integrals of local quantities. To apply (2.67) we have to write the fiducial Hamiltonian in the form (2.27) by using the conformal transformation (2.26). Because the theory is conformally invariant we get[13]

$$\bar{H}^2(\lambda) = -\bar{h}^{ij}(\bar{\nabla}_i + i\lambda a_i)(\bar{\nabla}_j + i\lambda a_j) + \frac{1}{8}\bar{R} + \frac{1}{32}F^{ik}F_{ik}, \tag{2.77}$$

[12] For instance, one can define for this purpose the zeta function $\zeta(\nu) = \sum_{nl} \tilde{\omega}_{nl}^{-2\nu}$ and get $c_0$ as the limit $2\pi(\nu - 1/2)\zeta(\nu)$ at $\nu = 1/2$ and $c_1$ as a limit $-4\pi(\nu + 1/2)\zeta(\nu)$ at $\nu = -1/2$. These limits can be computed directly from the series.

[13] To get (2.77) one has to take into account that scalar curvature of 3D space with metric $B^{-1}ds^2$ where $ds^2$ is metric (2.72) is $\bar{R} + \frac{1}{4}F^{ik}F_{ik}$.

where $a_i dx^i = a_\varphi d\tilde{\varphi}$, $\bar{\nabla}_i$ and $\bar{R}$ are covariant derivatives and the scalar curvature on the 2-dimensional space $\bar{\mathcal{B}}$ with the metric (see (2.73))

$$dl^2 = \frac{\sin^2\theta}{B^2} d\tilde{\varphi}^2 + \frac{1}{B} d\theta^2. \tag{2.78}$$

$B$ and $a_\varphi$ are defined in (2.74). For the operator (2.77) we get

$$c_0 = \int_{\bar{B}} \bar{h}^{1/2} d^2x = \text{Vol } \bar{\mathcal{B}}, \tag{2.79}$$

$$c_1 = \int_{\bar{B}} \bar{h}^{1/2} d^2x \left[ \frac{1}{24} \bar{R} - \frac{1}{32} \bar{F}^{ij} \bar{F}_{ij} \right], \tag{2.80}$$

where $\bar{F}_{ij} = F_{ij}$ is the Maxwell tensor for the fiducial potential $a_i$. By using (2.74) one can check that (2.79) and (2.80) coincide with (2.76). In the same way one can make the direct check of (2.67) for a conformal field in a rotating 4-dimensional Einstein universe.

## 2.5 High temperature limit

Let us consider a Gibbs state in a stationary space-time in the limit of high temperatures[14]. This limit is one of the most interesting and well studied regimes. When the parameter $\beta$ is small the dominant contribution to the free energy (1.6) results from large frequencies $\omega \simeq \beta^{-1}$. In this situation one can go from summation in (1.9) to the integral

$$F^C(\beta) = \beta^{-1} \int_\mu^\infty \Phi(\omega) d\omega \ln\left(1 - e^{-\beta\omega}\right), \tag{2.81}$$

where the spectral density $\Phi(\omega)$ should be determined by its asymptotic expansion (2.66). This gives

$$F^C(\beta) \sim -\frac{1}{\pi^{D/2}\beta^D} \sum_{n=0}^{\infty} \Gamma\left(\frac{D-2n}{2}\right) \zeta(D-2n) c_n \left(\frac{\beta}{2}\right)^{2n}. \tag{2.82}$$

Here $\zeta(x)$ is the Riemann zeta-function. High temperature asymptotics of this form have been obtained and studied on static space-times by Dowker and collaborators [14], [37], [38] (see also [39]). Equation (2.82) is the extension of these results to non-static case.

The following remarks regarding (2.82) are in order. First, formula (2.82) is purely local. However, the method we used to get (2.82) does not take into account a non-local term proportional to $\beta^{-1}$, see [14], [37]. This contribution comes for Bose fields from the region of small $\omega$ where the logarithm under the integral in (2.81) results in a term $\ln(\omega)$. Second, (2.82) is obtained with the help of dimensional regularization. When the parameter $D$ coincides with the physical dimension one of the terms in (2.82) has a simple pole. However if the system is in a finite box or there is a mass gap $\mu > 0$ this

[14] Certainly, there must exist necessary physical conditions which ensure the thermal equilibrium. These conditions require special analysis in each particular case. We just assume that in the problems considered here they are satisfied.

singularity is an artifact of the method. Careful analysis of the infrared singularities has been carried out recently by Gusev and Zelnikov [40] who have derived an approximate non-local expression for the one-loop free energy by using covariant perturbation theory of Ref. [41]. Further discussion of this problem can be found in [40].

In four dimensions ($D = 4$)

$$F^C(\beta) \sim -\frac{\pi^2}{90}\frac{c_0}{\beta^4} - \frac{1}{24}\frac{c_1}{\beta^2} - \frac{1}{16\pi^2}\ln(\beta\rho)c_2 - \frac{1}{16\pi^{5/2}}\sum_{n=3}^{\infty}\Gamma\left(n-\frac{3}{2}\right)\zeta(2n-3)c_n\left(\frac{\beta}{2\pi}\right)^{2n-4}, \tag{2.83}$$

where $\rho$ is a dimensional parameter related to the regularization and the pole term is omitted. When the frame does not rotate (2.83) coincides with the result of Ref. [14]. Of special interest are the leading terms in (2.83)

$$F^C(\beta) \sim -\int d^3x\sqrt{-g}\left[\frac{\pi^2}{90}T^4 + \frac{1}{24}T^2\left(\frac{1}{6}R - V - \frac{2}{3}\Omega^2\right)\right] - \frac{c_2}{16\pi^2}\ln(\beta\rho). \tag{2.84}$$

Here $T$ is the local temperature $T = 1/(\beta\sqrt{B})$ and $R$ is the scalar curvature of the background space-time. The second term in r.h.s of (2.84) depends on the local angular velocity $\Omega$ of the system with respect to a local Lorentz frame, see (2.12). This result follows from the form of the $c_1$ coefficient (2.69) which can be rewritten in terms of the characteristics of the physical space-time as [34]

$$c_1 = \int \sqrt{-g}d^3x\frac{1}{B}\left[\frac{1}{6}R - V - \frac{2}{3}\Omega^2\right]. \tag{2.85}$$

As for $c_2$ in (2.84), we will show in section 3.5 that it does not depend on the rotation and has a standard covariant form quadratic in four-dimensional curvatures.

The validity of (2.84) can be checked for some simple models where the spectrum is known. The simplest case is again conformal fields in rotating Einstein universe. Such models have been recently investigated by several authors [42]–[45]. The formula (2.84) is in agreement with these results.

## 2.6 Thermal part of the stress-energy tensor

The free energy $F^C(\beta)$ of a Gibbs state on a stationary space-time is a functional of the background metric $g_{\mu\nu}$ and the Killing vector $\xi$. It is invariant with respect to time independent coordinate transformations because these transformations do not change the form of the metric, see (2.6)–(2.9). Infinitesimal changes of coordinates can be written as

$$\delta x^\mu = \zeta^\mu(x^i), \tag{2.86}$$

where $\mu$ runs from 1 to $D$ and $i$ from 1 to $D-1$. The condition that coordinate changes are time independent can written as

$$(\mathcal{L}_\xi\zeta)^\mu = -(\mathcal{L}_\zeta\xi)^\mu = 0, \tag{2.87}$$

where $\xi = \partial_t$ is the corresponding Killing field, and $\mathcal{L}_\xi$ is the Lie derivative along $\xi$. Thus, (2.87) is equivalent to the condition that the Killing field is left unchanged under transformations generated by $\zeta^\mu(x^i)$. In a stationary space-time the group of transformations is $D$-dimensional and by the Noether theorem the invariance of $F^C(\beta)$ implies existence of $D$ Noether currents and corresponding charges. Like in a classical theory, the Noether current can be defined as $T_{\mu\nu}\xi^\nu$ where $T_{\mu\nu}$ is the stress-energy tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \left(\frac{\delta F^C(\beta)}{\delta g_{\mu\nu}}\right)_\xi . \tag{2.88}$$

According to (2.87) the variations are to be taken at fixed $\xi$ [15]. Note that $F(\beta)$ is a functional in a $D-1$ dimensional space, so $g_{\mu\nu}$ are functions of $D-1$ coordinates. One can use (2.84) to find $T^{\mu\nu}$ in high-temperature limit As an illustration, we present the result for conformally coupled scalar field ($V = R/6$ in (2.84))

$$T_{\mu\nu} \sim (g_{\mu\nu} + 4u_\mu u_\nu)\left(\frac{\pi^2}{90}T^4 - \frac{1}{36}T^2\Omega^2\right)$$

$$+\frac{T^2}{36}\left(2A_{\nu\rho}A_\mu{}^\rho - u_\nu A_{\mu\lambda}w^\lambda - u_\mu A_{\nu\lambda}w^\lambda + u_\nu A_{\mu\lambda}{}^{;\lambda} + u_\mu A_{\nu\lambda}{}^{;\lambda}\right) + O(\ln T). \tag{2.89}$$

Here $u^\mu$ is the velocity vector of the Killing frame where thermal equilibrium is defined, $w_\mu$ is its acceleration and $A_{\mu\nu}$ is the antisymmetric rotation tensor (2.5). The invariance of $F(\beta)$ implies the standard covariant conservation law $T^{\mu\nu}{}_{;\nu} = 0$. Indeed, by considering $F(\beta)$ as local functional (2.84) one can always write the change of $F^C(\beta)$ under coordinate transformations as

$$\delta_\zeta F^C(\beta) = \int d^3x\sqrt{-g}\left[\nabla_\mu j^\mu(\zeta, \delta_\zeta g) - T^{\mu\nu}\zeta_{\mu;\nu}\right], \tag{2.90}$$

where $j_\mu$ is a local expression of $\zeta$ and variations of the metric. Because the background theory is stationary the divergence of $j^\mu$ reduces to three-dimensional divergence which results in a surface term. For stationary transformations (2.86) the expression (2.90) vanishes which is possible only when $T^{\mu\nu}$ is divergence free. This justifies its interpretation as a stress-energy tensor.

By taking into account that $u_\mu A^{\mu\nu}{}_{;\nu} = -A_{\mu\nu}A^{\mu\nu}$ one can check that $T^\mu_\mu = 0$. Hence (2.89) is interpreted as a stress-energy tensor of a massless conformally invariant matter with temperature $T$ and velocity $u_\mu$. The first term of (2.89) coincides with the stress-tensor of an ideal gas at temperature $T$.

It should be noted that (2.88) represents the average value of the thermal part of a quantum stress tensor-energy tensor in the Gibbs state. It does not coincide, however, with the total stress-energy tensor because (2.88) does not include the vacuum part[16].

[15]In practice one has to keep fixed contravariant components $\xi^\mu$ of the Killing field. For instance, variation of $B = -\xi^2$ then looks as $(\delta B)_\xi = -\xi^\mu\xi^\nu\delta g_{\mu\nu}$.

[16]Let us recall that the contribution of zero-point energies in $F^C(\beta)$ has been eliminated from the very beginning by normal ordering, see section 1.1.

Because the conformal anomaly is related to renormalization only of the vacuum energy this fact explains why (2.89) is traceless.

High-temperature asymptotics like (2.89) may represent an interest in studying quantum effects near rotating black holes because the local temperature $T$ becomes large near the black hole horizon. To realize such a state for a Kerr–Newman black hole one has to place it in a cavity which co-rotates rigidly with the black hole. Hence, for Kerr-Newman black hole (2.89) requires to take into account effects of the cavity. The situation is simpler for rotating black holes with asymptotic anti-de Sitter geometry (Kerr–AdS black holes [27]). They are solutions in gravity theory with negative cosmological constant. If the angular velocity of Kerr-AdS black hole is less than inverse AdS radius the Killing vector field $\xi$ is globally time-like outside the horizon. The cavity is not necessary in this case and (2.89) may be a correct approximation to the thermal part of stress-energy tensor, at least near the horizon.

# 3 Euclidean theory

## 3.1 Formulation of the problem

Let us consider a stationary space-time $\mathcal{M}$ with metric defined by (2.6)–(2.9). Suppose that the metric depends on a number of parameters $J$ denoted by a collective symbol $J$. These parameters will play the same role as the angular velocity in the metric (1.17). We assume that there is a complexification of $\mathcal{M}$ (in the sense described in section 1.2) which has a real Euclidean section $\mathcal{M}_E$ with metric

$$ds_E^2 = \breve{B}(d\tau + \breve{a}_k dx^k)^2 + \breve{h}_{kj} dx^k dx^j. \tag{3.1}$$

$\tau$ is a periodic coordinate with period $\beta$. We also assume that (3.1) can be obtained from (2.6) by replacement $t = -i\tau$ and analytical continuation of the components to imaginary values of $J$

$$\breve{h}_{kj}(x;J) = h_{kj}(x;iJ)\ ,$$
$$\breve{a}_k(x;J) = i a_k(x;iJ)\ ,\ \ \breve{B}(x;J) = B(x;iJ)\ . \tag{3.2}$$

An example of this procedure has been discussed in section 1.2. The Euclidean effective action $W^E(\beta, J)$ for free fields or in one-loop approximation is a determinant of the corresponding Euclidean operators $L_E$, see (1.10). In what follows we consider the action defined with the help of $\zeta$ function regularization [46],[47] as

$$W^E(\beta, J) = -\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta(\nu|\beta, J), \tag{3.3}$$

$$\zeta(\nu|\beta, J) = \sum_{\Lambda}\left[\varrho^2\Lambda\right]^{-\nu}, \tag{3.4}$$

where $\Lambda$ are eigen-values of $L_E$, $\varrho$ is a dimensional parameter related to regularization. The series in (3.14) converge for $Re\ \nu > D/2 - 1$. To get $\zeta$ function for other values of $\nu$ one considers analytical continuation. It can be shown that $\zeta(\nu|\beta)$ is a meromorphic function of $\nu$ with simple poles on the real axis. In particular, this function is regular near $\nu = 0$ so that one can use (3.3).

To go to the physical space-time one has to analytically continue (3.3) back to some real values of $J$ (these values are to be in an interval corresponding to the rotation with the velocity smaller than the velocity of light). This results in a new functional

$$\beta F^E(\beta) = W^E(\beta, -iJ). \tag{3.5}$$

We call $F^E(\beta)$ the Euclidean free energy. Our purpose now is to describe the relation between $F^E(\beta)$ and canonical free energy $F^C(\beta)$, Eq. (1.9)

As earlier we consider the scalar model (2.18). Then $\zeta$-function (3.4) is determined by the eigen-values of the Euclidean wave operator on $\mathcal{M}_E$

$$(-\nabla^\mu\nabla_\mu + V)\phi_\Lambda = \Lambda\phi_\Lambda. \tag{3.6}$$

The Kaluza–Klein method can be applied to Euclidean theory as follows. Note that because of the isometry, $\phi_\Lambda$ are eigen-vectors of the operator $i\partial_\tau$

$$\phi_\Lambda(\tau, x) = e^{i\sigma_l\tau}\phi_\Lambda(x), \tag{3.7}$$

where $\sigma_l = (2\pi l)/\beta$, $l = 0, \pm1, \pm2, ....$ For these modes (3.6) takes the form

$$(-\breve{g}_E^{\mu\nu}(\breve{\nabla}_\mu + i\sigma_l\breve{a}_\mu)(\breve{\nabla}_\nu + i\sigma_l\breve{a}_\nu) + V)\phi_\Lambda = \Lambda\phi_\Lambda. \tag{3.8}$$

Here the metric and the connections correspond to a fiducial static Euclidean space-time $\tilde{\mathcal{M}}_E$

$$d\breve{s}_E^2 = (\breve{g}_E)_{\mu\nu}dx^\mu dx^\nu = \breve{B}d\tau^2 + \breve{h}_{jk}dx^j dx^k. \tag{3.9}$$

Thus, the Euclidean problem on a stationary background can be reformulated as a theory on a static background in the presence of a fiducial gauge connection. The distinction from the Lorentzian theory is that the fiducial charges $\sigma_l$ (Matsubara frequencies) are quantized because the Euclidean time coordinate $\tau$ is compact.

Without loss of generality we again suppose that $\breve{B} = 1$ in (3.9). One can always use conformal transformation to bring the metric to this form. The result of this transformation is an anomalous addition to the Euclidean action. This addition is proportional to $\beta$ and changes only the vacuum energy. We will discuss this question in section 3.4. If $\breve{B} = 1$ the eigen-value problem can be written as

$$(\sigma_l^2 + \breve{H}^2(\sigma_l))\phi_\Lambda = \Lambda\phi_\Lambda, \tag{3.10}$$

$$\breve{H}^2(\sigma_l) = -\breve{h}^{jk}(\breve{\nabla}_j + i\sigma_l\breve{a}_j)(\breve{\nabla}_k + i\sigma_l\breve{a}_k) + V. \tag{3.11}$$

The operators $\breve{H}(\sigma_l)$ are analogous to the Lorentzian Hamiltonian $H(\lambda)$. $\breve{H}^2(\sigma_l)$ are positive-definite elliptic operators.

Let us consider first systems on a compact space. The spectrum of $\breve{H}^2(\sigma_l)$ is discrete and bounded from below and we suppose it is strictly positive, for simplicity. Denote the corresponding eigen-values by $\breve{\omega}^2(\sigma_l)$. Given solutions to the problem

$$\breve{H}^2(\sigma_l)\phi_\omega^{(\sigma_l)}(x^i) = \breve{\omega}^2(\sigma_l)\phi_\omega^{(\sigma_l)}(x^i), \tag{3.12}$$

the solution to (3.6) can be represented as

$$\phi_\Lambda(\tau, x) = e^{i\sigma_l\tau}\phi_\omega^{(\sigma_l)}(x^i), \quad \Lambda = \sigma_l^2 + \breve{\omega}^2(\sigma_l). \tag{3.13}$$

Then $\zeta$-function (3.4) is

$$\zeta(\nu|\beta) = \varrho^{-2\nu}\sum_{\sigma_l}\sum_{\omega}(\sigma_l^2 + \breve{\omega}^2(\sigma_l))^{-\nu} \quad , \tag{3.14}$$

where the sum is taken over all eigen-values $\breve{\omega}^2(\sigma_l)$ of $\breve{H}^2(\sigma_l)$ (some eigen-values may coincide). The Euclidean effective action is defined with the help of (3.14) by (3.3).

### 3.2 From Euclidean to Lorentzian theory

By analogy with the Lorentzian theory, where fiducial charges are continuous, let us consider a one-parameter family of operators $\breve{H}^2(\lambda)$ where the parameter $\lambda$ is real. For the considered problem $\breve{H}^2(\lambda)$ are positive-definite elliptic operators with eigen-values $\breve{\omega}^2(\lambda)$.

We make the following additional assumptions:

1) The Wick rotation to Lorentzian theory is determined by analytical continuation of a set of parameters $J$ in such a way that under this continuation eigen-values of $\breve{H}^2(\lambda)$ transform into eigen-values of the Lorentzian operators $H^2(\lambda)$, i.e.

$$\breve{\omega}^2(i\lambda| - iJ) = \omega^2(\lambda|J). \tag{3.15}$$

Equation (3.15) should be considered simply as a condition on $\breve{\omega}^2$ as functions of $\lambda$ and $J$.

2) As in the Lorentzian theory, see (2.32), we define functions $\breve{\chi}(\breve{\omega}, \lambda) = \breve{\omega}^2(\lambda) + \lambda^2$. For $\lambda = \sigma_l$ they coincide with eigen-values $\Lambda$ of the Euclidean operartor $L_E$. We assume that $\breve{\chi}(\omega, z)^{-1}$ can be analytically continued in $z$ to the upper complex plane where they are meromorphic functions with simple poles. Note that according to (3.15), $\breve{\chi}(\breve{\omega}, \lambda)$ transform to $-\chi(\omega, \lambda)$, functions defined in (2.32). Thus, after the Wick rotation the poles lie on axes $Re\ z = 0$ and coincide with physical energies. The requirement that the poles are simple is equivalent to condition $\breve{\chi}'(\omega, i\omega) \neq 0$, where a prime denotes the derivative with respect to $\lambda$. This is in accord with condition that physical states have non-zero norm, see (2.44).

For simple models one can check that our second assumption about $\breve{\chi}(\breve{\omega},\lambda)$ does hold true. For instance, for a field in a rotating Einstein universe (discussed in section 2.4) the eigen values of the Euclidean operator $L_E = -\nabla^2 + R/8$ are $\Lambda = (\sigma_k + l\breve{\Omega}_0)^2 + \omega_n^2$ where $\breve{\Omega}_0$ is the Euclidean angular velocity. Hence, $\breve{\chi}(\breve{\omega},\lambda) = (\lambda + l\breve{\Omega}_0)^2 + \omega_n^2$ and under change $\breve{\Omega}_0$ to $-i\Omega_0$ one gets $\breve{\chi}(\breve{\omega}, i\lambda) = -(\lambda - l\Omega_0)^2 + \omega_n^2$. They are the functions whose zeros $\lambda = \pm(\omega_n + l\Omega_0)$ coincide with physical energies. Also, $\breve{\chi}'(\omega, i\omega) = 2i\omega_n \neq 0$.

We now follow [48] and rewrite the $\zeta$-function by using the Cauchy theorem as

$$\zeta(\nu|\beta) = \frac{\varrho^{-2\nu}}{2\pi i} \sum_{\sigma_l} \sum_{\omega} \int_C \frac{dz}{z - \sigma_l} (z^2 + \breve{\omega}^2(z))^{-\nu}. \tag{3.16}$$

The contour $C$ consists of two parallel lines, $C_+$ and $C_-$, in the complex plane. $C_+$ goes from $(i\epsilon + \infty)$ to $(i\epsilon - \infty)$ and $C_-$ goes from $(-i\epsilon - \infty)$ to $(-i\epsilon + \infty)$. Here $\epsilon$ is a small positive parameter such that $\epsilon < \mu$. We consider (3.16) at $Re\ \nu > D/2 - 1$ and assume that $D \geq 2$. Summation over $\sigma_l$ in (3.16) can be performed

$$\zeta(\nu|\beta) = \frac{\varrho^{-2\nu}\beta}{4\pi i} \sum_{\omega} \int_C dz \cot\left(\frac{\beta z}{2}\right) (z^2 + \breve{\omega}^2(z))^{-\nu}. \tag{3.17}$$

Note that spectrum $\breve{\omega}^2(\lambda)$ has to be symmetric with respect to change $\lambda$ to $-\lambda$. Hence, integrations over $C_+$ and $C_-$ in (3.17) coincide and one can write (3.17) as twice the integral over $C_+$. To proceed let us use the identity

$$\cot\left(\frac{\beta z}{2}\right) = \frac{2}{\beta}\frac{d}{dz} \ln\left(1 - e^{i\beta z}\right) - i$$

which enables one to write

$$\zeta(\nu|\beta) = \beta\zeta_0(\nu) + \zeta_T(\nu|\beta), \tag{3.18}$$

$$\zeta_0(\nu) = \frac{\varrho^{-2\nu}}{\pi} \sum_{\omega} \int_0^{\infty} (x^2 + \breve{\omega}^2(x))^{-\nu} dx, \tag{3.19}$$

$$\zeta_T(\nu|\beta) = \frac{\varrho^{-2\nu}}{\pi i} \sum_{\omega} \int_{C_+} dz \frac{d}{dz} \ln\left(1 - e^{i\beta z}\right) (z^2 + \breve{\omega}^2(z))^{-\nu}. \tag{3.20}$$

$\zeta_T(\nu|\beta)$ represents the purely thermal part which vanishes at zero temperature because of small positive imaginary part of $z$ in $e^{i\beta z}$. This means that the only quantity which can be responsible for the vacuum energy is $\zeta_0(\nu)$. Let us now deform $C_+$ in (3.19) so as to make the integrand exponentially small at large $z$ due to the factor $e^{i\beta z}$. After that we can integrate by parts to get

$$\zeta_T(\nu|\beta) = \nu\frac{\varrho^{-2\nu}}{\pi i} \sum_{\omega} \int_{C_+} dz \ln\left(1 - e^{i\beta z}\right) \frac{\breve{\chi}'(\breve{\omega}, z)}{(z^2 + \breve{\omega}^2(z))^{\nu+1}}. \tag{3.21}$$

We can now represent the Euclidean action in the following form (see (3.3), (3.18))

$$W^E(\beta) = \beta(\breve{F}(\beta) + \breve{E}_0), \tag{3.22}$$

$$\breve{F}(\beta) = -\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta_T(\nu|\beta), \tag{3.23}$$

$$\breve{E}_0 = -\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta_0(\nu), \tag{3.24}$$

where $\zeta_T(\nu|\beta)$ and $\zeta_0(\nu)$ are given by (3.19) and (3.21), respectively. To compute $\breve{F}(\beta)$ we note that $\zeta_T(\nu|\beta)$ has a form $\zeta(\nu|\beta) = \nu f(\nu|\beta)$. To find $f(\nu|\beta)$ at $\nu = 0$ we add to $C_+$ a large semicircle lying in the upper half of the complex plane to make a closed contour. Because of the exponent $e^{i\beta z}$ in the logarithm in (3.21) the integration over the semicircle vanishes when its radius goes to infinity. Our assumption guarantees that the function $f(\nu|\beta)$ is finite at $\nu = 0$ and by the Cauchy theorem its value is determined by the residues at $z = i\breve{\omega}$. We get

$$\breve{F}(\beta) = \frac{1}{\beta}\sum_z \ln\left(1 - e^{-\beta z}\right), \tag{3.25}$$

where $z$ are corresponding solutions to $\breve{\chi}(\breve{\omega}, iz) = 0$. Finally, the first assumption guarantees that after the Wick rotation (3.25) coincides with the canonical free energy

$$F^C(\beta) = \frac{1}{\beta}\sum_\omega \ln\left(1 - e^{-\beta\omega}\right), \tag{3.26}$$

where the sum is taken over all non-negative solutions of the equation $\omega^2(z) = z^2$ (see (2.31), (2.32)). Thus, according to (3.5), (3.22) the canonical and Euclidean free energies are related by Eq. (1.16) where $E_0$ coincides with $\breve{E}_0$ after the Wick rotation. We will see in section 3.4 that $\breve{E}_0$ does correspond to the vacuum energy.

## 3.3 Continuous spectrum

It is instructive to discuss what happens in case when the spectrum is continuous. The Euclidean theory offers another look at the origin of key relations (2.56), (2.60) which determine the physical spectral density $\Phi(\omega)$.

The Euclidean action is defined by (3.3) where the $\zeta$ function now has the form

$$\zeta(\nu|\beta) = \varrho^{-2\nu}\int_\mu^\infty d\omega \sum_{\sigma_l}\breve{\Phi}(\omega;\sigma_l)(\sigma_l^2 + \omega^2)^{-\nu}. \tag{3.27}$$

We assume that $\breve{H}^2(\sigma_l)$ have a positive mass gap $\mu$ which does not depend on Matsubara frequencies $\sigma_l$. Similarly to the discrete case, the $\zeta$-function can be rewritten by using the Cauchy theorem in (3.18) where

$$\zeta_0(\nu) = \frac{\varrho^{-2\nu}}{\pi}\int_\mu^\infty d\omega\int_0^\infty \breve{\Phi}(\omega;x)(x^2+\omega^2)^{-\nu}dx, \tag{3.28}$$

$$\zeta_T(\nu|\beta) = \frac{\varrho^{-2\nu}}{\pi i}\int_\mu^\infty d\omega\int_{C_+} dz\frac{d}{dz}\ln\left(1 - e^{i\beta z}\right)\breve{\Phi}(\omega;z)(z^2+\omega^2)^{-\nu}, \tag{3.29}$$

and $C_+$ is defined as earlier. Then the Euclidean action is represented in the form (3.22) where $\breve{F}(\beta)$ and $\breve{E}_0$ are defined with the help of (3.28) and (3.29), respectively.

To compute $\zeta_T(\nu|\beta)$ we proceed as before and deform $C_+$ in (3.29) so as to make the integrand exponentially small at large $z$ due to the factor $e^{i\beta z}$ [17]. Integration by parts over $z$ then gives

$$\zeta_T(\nu|\beta) = \frac{\varrho^{-2\nu}}{\pi i} \int_\mu^\infty d\omega \int_{C_+} dz \ln\left(1 - e^{i\beta z}\right) \left[\frac{2\nu z \breve{\Phi}(\omega; z)}{(z^2+\omega^2)^{\nu+1}} - \frac{\partial_z \breve{\Phi}(\omega; z)}{(z^2+\omega^2)^\nu}\right]. \tag{3.30}$$

One can also integrate by parts over $\omega$ the second term in square brackets[18]

$$\zeta_T(\nu|\beta) = \frac{\nu \varrho^{-2\nu}}{\pi i} \int_\mu^\infty d\omega \int_{C_+} dz \frac{2z}{(z^2+\omega^2)^{\nu+1}} \ln\left(1 - e^{i\beta z}\right) \breve{\Psi}(\omega; z), \tag{3.31}$$

$$\breve{\Psi}(\omega; z) = \breve{\Phi}(\omega; z) - \frac{\omega}{z} \int_\mu^\omega \frac{d}{dz} \breve{\Phi}(\sigma; z) d\sigma. \tag{3.32}$$

Function $\breve{\Psi}(\omega; z)$ corresponds to auxiliary density $\Psi(\omega, \lambda)$, see (2.60). In the Lorentzian theory $\Psi(\omega, \lambda)$ appears as result of analysis of the inner products. It is remarkable that in the Euclidean theory there appears an analogous function $\breve{\Psi}(\omega; z)$, although it happens on a completely different footing.

To compute $\breve{F}(\beta)$ one can use the fact that $\zeta_T(\nu|\beta)$ now has a form $\zeta(\nu|\beta) = \nu f(\nu|\beta)$, see (3.31). We follow [48] and assume that for any $\omega$ the density $\breve{\Psi}(\omega, z)$ can be analytically continued to complex $z$ and is an entire function of $z$ in the upper half of the complex plane. One can then proceed as in section 3.2 and compute $f(\nu|\beta)$ at $\nu = 0$ by using the Cauchy theorem. The result looks as follows:

$$\breve{F}(\beta) = \frac{1}{\beta} \int_\mu^\infty d\omega \breve{\Phi}(\omega) \ln\left(1 - e^{-\beta\omega}\right), \tag{3.33}$$

$$\breve{\Phi}(\omega) = \left.\breve{\Psi}(\omega; z)\right|_{z=i\omega}. \tag{3.34}$$

Similarity between $\breve{\Phi}(\omega)$ and physical density $\Phi(\omega)$ defined by (2.56), (2.60) is obvious. Consider now the Wick rotation from Euclidean $\mathcal{M}_E$ to Lorentzian space-time $\mathcal{M}$ determined by analytical continuation of a set of the parameters $J$ of the metric. Our second assumption is that $\breve{\Psi}(\omega; \lambda)$, as a function of parameters $J$, can be analytically continued to complex $J$ in such a way that the following equality holds:

$$\breve{\Psi}(\omega; i\lambda| - iJ) = \Psi(\omega; \lambda|J) \tag{3.35}$$

for $J$ and $\lambda$ real. Condition (3.35) is an analog of (3.15). Given equations (3.33)–(3.35) one concludes that $\breve{\Phi}(\omega)$ and $\breve{F}(\beta)$ coincide with the physical density of levels and the free energy, respectively,

$$\breve{\Phi}(\omega| - iJ) = \Phi(\omega|J), \tag{3.36}$$

[17] We conjecture that $\breve{\Phi}(\omega; z)$ itself cannot increase at large $z$ and presence of the factor $e^{i\beta z}$ is enough to ensure convergence of (3.29).

[18] The boundary terms vanish at Re $\nu > D/2 - 1$ because $\partial_z \breve{\Phi}(\omega; z) \sim \omega^{D-4}$ at large $\omega$.

$$\breve{F}(\beta|-iJ)=F(\beta|J). \tag{3.37}$$

A rigorous proof of (3.35) is problematic because the explicit form of $\Phi(\omega;\lambda)$ is not known in general. One can demonstrate [48], however, that this assumption is true at least asymptotically in the limit of high frequencies when the spectral density becomes a local functional of the background geometry and the Wick rotation procedure can be easily applied.

## 3.4 Vacuum energy

Let us return to the vacuum part $\breve{E}_0$ of the Euclidean action, see (3.24). According to (3.19),

$$\breve{E}_0=-\frac{1}{2}\lim_{\nu\to 0}\frac{d}{d\nu}\zeta_0(\nu), \tag{3.38}$$

$$\zeta_0(\nu)=\frac{1}{\pi\varrho}\int_0^\infty dx\zeta\left(\nu|\varrho^2(\breve{H}^2(x)+x^2)\right), \tag{3.39}$$

where $\zeta(\nu|\mathcal{O})$ is the generalized $\zeta$-function of an operator $\mathcal{O}$. Let us show that after the Wick rotation $\breve{E}_0$ agrees with the standard definition of the vacuum energy. For this purpose we represent (3.19) in the form

$$\zeta_0(\nu)=-\frac{\varrho^{-2\nu}}{2\pi}\sum_\omega\int_{C_+}dz(z^2+\breve{\omega}^2(z))^{-\nu}e^{i\epsilon z}, \tag{3.40}$$

where a small positive parameter $\epsilon$ is introduced to regularize the integral in the limit of vanishing $\nu$. By integrating in (3.40) by parts over $z$, and neglecting terms linear in $\epsilon$, we get

$$\zeta_0(\nu)=-\nu\frac{\varrho^{-2\nu}}{2\pi}\sum_\omega\int_{C_+}dz\frac{z\breve{\chi}'(\breve{\omega},z)}{(z^2+\breve{\omega}^2(z))^{\nu+1}}e^{i\epsilon z}. \tag{3.41}$$

The regularization enables us to replace $C_+$ by a closed contour in the upper half of complex plane and use the Cauchy theorem

$$\breve{E}_0=\frac{1}{2}\sum_z ze^{-z}, \tag{3.42}$$

where $z$ are solutions to $\breve{\chi}(\breve{\omega},iz)=0$. After the Wick rotation $\breve{E}_0$ does correspond to the vacuum energy determined with cutoff $1/\epsilon$ in the range of high frequencies.

A comment regarding relations (3.22) and (3.42) is in order. These relations are valid when the Killing field $\xi$ has a unit norm, i.e., the space-time is "ultrastationary" with metric

$$ds^2=(d\tau+\breve{a}_kdx^k)^2+\breve{h}_{kj}dx^kdx^j, \tag{3.43}$$

where $\tau$ is periodic with period $\beta$. One can always use the conformal rescaling $\bar{g}_{\mu\nu}=g_{\mu\nu}/B$ to reduce the metric $g_{\mu\nu}$, see (3.1), to ultrastationary form (3.43). The (renormalized) Euclidean action $W^E[g]$ on (3.1) is related to the action $\bar{W}^E[\bar{g}]$ on (3.43) as

$$W^E[g]=\bar{W}^E[\bar{g}]+\beta\Delta[g], \tag{3.44}$$

where $\beta\Delta[g]$ appears as a result of the conformal anomaly[19]. It can be shown that $\Delta[g]$ is a local functional [37],[38]. According to (3.43), the vacuum energy computed for an arbitrary stationary space-time is $E_0[\bar{g}]+\Delta[g]$, where $E_0[\bar{g}]$ is defined by (3.38), (3.39) on on (3.43)

Equations (3.38), (3.39) hold for discrete and continuous spectra. They enable one to compute the vacuum energy of fields in arbitrary stationary space-times by using the $\zeta$-function method and can be used in different applications.

## 3.5 Kaluza-Klein reduction of heat-kernel coefficients

Let us discuss now the functions $\zeta(\nu|\beta)$, $\zeta_0(\nu)$ and relation (3.18) (which holds on ultrastationary space-times) in more detail. We begin with $\zeta(\nu|\beta)=\zeta\left(\nu|\varrho^2 L\right)$ which is a generalized $\zeta$ function of the Euclidean operator $L_E=-\nabla^2+V$, see (3.6) and (3.14). According to the general theory, $\zeta(\nu|\beta)$ is a meromorphic function which has simple poles on the real axis of $\nu$. In the theory with $D$ dimensions the part $\zeta^{(p)}(\nu|\beta)$ which includes the poles of $\zeta(\nu|\beta)$ can be written as (see, e.g., [17])

$$\zeta^{(p)}(\nu|\beta)=\frac{2}{(4\pi)^{D/2}\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{\breve{A}_n}{2\nu+2n-D}, \tag{3.45}$$

where $\breve{A}_n$ are the coefficients related to the asymptotic expansion of the heat kernel[20]

$$\mathrm{Tr}e^{-tL_E}\sim\frac{1}{(4\pi t)^{D/2}}\sum_{n=0}^{\infty}\breve{A}_n t^n. \tag{3.46}$$

Relation (3.18) has an important consequence: the poles of $\zeta(\nu|\beta)$ coincide with poles of $\beta\zeta_0(\nu)$. This happens simply because coefficients $\breve{A}_n$ in (3.45) depend on parameter $\beta$ linearly. To investigate the poles of $\zeta_0(\nu)$ we rewrite (3.39) as

$$\zeta_0(\nu)=\frac{1}{\pi\Gamma(\nu)}\int_0^\infty dx\int_0^\infty dt\ t^{\nu-1}\mathrm{Tr}\ e^{-t(\breve{H}^2(x)+x^2)}. \tag{3.47}$$

Our assumption is that the poles of $\zeta_0$ are related to the behaviour of the integral at small $t$. The argument is that in this limit the trace is a local functional. By following [17] we define the pole part $\zeta_0^{(p)}(\nu)$ of $\zeta_0(\nu)$ by (3.47) with integration over $t$ taken in the interval $(0,1)$. One can then replace the trace of the heat kernel of $\breve{H}^2(x)$ by its asymptotic expansion

$$\mathrm{Tr}e^{-t\breve{H}^2(\lambda)}\simeq\frac{1}{(4\pi t)^{(D-1)/2}}\sum_{n=0}^{\infty}\breve{a}_n(\lambda)t^n, \tag{3.48}$$

[19] The anomalous term is related to breaking of conformal invariance in the process of renormalization of the effective action. The anomaly can be also attributed to non invariance of the integration measure in the Euclidean functional integral [49].

[20] In what follows we put $\varrho=1$ for simplicity. We will also assume that space-time has no boundaries and hence $n$ in (3.46) is an integer.

and get

$$\zeta_0^{(p)}(\nu) = \frac{1}{\pi\Gamma(\nu)} \int_0^\infty dx \int_0^1 dt\ t^{\nu-1} e^{-tx^2} \frac{1}{(4\pi t)^{(D-1)/2}} \sum_{n=0}^\infty \breve{a}_n(x) t^n. \tag{3.49}$$

The integral exists at Re $\nu > (D-1)/2$. To proceed we note that the heat kernel coefficients are polynomials analogous to (2.62)

$$\breve{a}_n(\lambda) = \sum_{m=0}^{[n/2]} \lambda^{2m} \breve{a}_{2m,n}, \tag{3.50}$$

where $\breve{a}_{2m,n}$ are some coefficients. It enables one to integrate (3.49) over $x$ and then over $t$

$$\zeta_0^{(p)}(\nu) = \frac{2}{(4\pi)^{D/2}\Gamma(\nu)} \sum_{n=0}^\infty \sum_{m=0}^{[n/2]} \frac{\Gamma(m+1/2)\breve{a}_{2m,n}}{\sqrt{\pi}(2\nu + 2(n-m) - D)}. \tag{3.51}$$

The latter equation can be rewritten also as

$$\zeta_0^{(p)}(\nu) = \frac{2}{(4\pi)^{D/2}\Gamma(\nu)} \sum_{n=0}^\infty \frac{1}{2\nu + 2n - D} \sum_{m=n}^{2n} \frac{\Gamma(m-n+1/2)}{\sqrt{\pi}} \breve{a}_{2(m-n),m}. \tag{3.52}$$

By comparing poles of (3.45) and (3.52) one concludes that

$$\breve{A}_n = \frac{\beta}{\sqrt{\pi}} \sum_{m=n}^{2n} \Gamma(m-n+1/2) \breve{a}_{2(m-n),m}. \tag{3.53}$$

For $D$ odd this conclusion is true for all $n$ while for $D$ even only for $0 \leq n \leq D/2 - 1$ (terms with other $n$ do not result in poles). It is clear, however, that (3.53) should be universal for all $D$ and all $n$ because dimensionality does not appear in it explicitly.

Equation (3.53) can be considered as a Kaluza–Klein reduction formula for heat kernel coefficients of an operator on a stationary $D$ dimensional manifold. It has one interesting and important consequence at $n = D/2$ for even $D$. Namely, quantity $\beta^{-1}\breve{A}_{D/2}$ transforms under the Wick rotation to the spectral coefficient $c_{D/2}$ defined by (2.67). This can be easily seen with the help of (3.53) if we take into account the relation between Euclidean and Lorentzian coefficients

$$\breve{a}_n(-iJ, i\lambda) = a_n(J, \lambda). \tag{3.54}$$

Equations (3.54), (2.62) and (3.50) imply that

$$\breve{a}_{2m,n}(-iJ) = (-1)^m a_{2m,n}(J). \tag{3.55}$$

The fact that $c_{D/2}$ is related to $\beta^{-1}\breve{A}_{D/2}$ is remarkable for two reasons. First, because $\breve{A}_{D/2} = (4\pi)^{D/2}\zeta(0|\beta)$ and $\zeta(0|\beta)$ determines the anomalous scaling of the Euclidean effective action [35]. Second, because this fact implies that $c_{D/2}$ is a local covariant functional of $D$-dimensional curvatures. One can, thus, conclude that the coefficient $c_2$ which appears in high temperature asymptotic (2.84) in four dimensions is a covariant functional quadratic in curvatures, the same property it has in static space-times.

The argument we used to derive (3.53) is not quite rigorous. Relation (3.53), however, passes a non-trivial check. Consider (3.53) for $n \neq 0$ on a $D$-dimensional stationary Euclidean background (3.43) where $\tau$ is periodic with period $\beta$. We denote this space by $\mathcal{M}_E$. The $D-1$-dimensional space $\mathcal{B}$ corresponding to operators $\breve{H}^2(x)$ is determined by the metric

$$dl^2 = h_{kj} dx^k dx^j . \tag{3.56}$$

We denote the Riemann tensors on $\mathcal{M}_E$ and $\mathcal{B}$ as $\breve{R}_{\mu\nu\lambda\rho}$ and $\bar{R}_{\mu\nu\lambda\rho}$, respectively. Their relation is discussed in Ref. [48].

Consider (3.53) for $n = 1$. It can be shown [48] that in any dimension

$$\breve{A}_1 = \beta \int \sqrt{h} d^{D-1}x \left(\frac{1}{6}\breve{R} - V\right) = \beta \int \sqrt{h} d^{D-1}x \left(\frac{1}{6}\bar{R} - \frac{1}{24}F^{\alpha\beta}F_{\alpha\beta} - V\right), \tag{3.57}$$

where the last line holds on ultrastationary space-times. On the other hand, according to (3.53)

$$\breve{A}_1 = \beta \left(\breve{a}_{0,1} + \frac{1}{2}\breve{a}_{2,2}\right). \tag{3.58}$$

By using heat kernel asymptotics in external gauge field one finds

$$\breve{a}_{0,1} = \int \sqrt{h} d^{D-1}x \left(\frac{1}{6}\bar{R} - V\right), \tag{3.59}$$

$$\breve{a}_{2,2} = -\int \sqrt{h} d^{D-1}x \frac{1}{12} F^{\alpha\beta}F_{\alpha\beta}. \tag{3.60}$$

Hence, the right hand sides of (3.57) and (3.58) do coincide. Equation (3.53) can be also checked for the coefficient $\breve{A}_2$, which on a manifold without boundary looks as

$$\breve{A}_2 = \beta \int \sqrt{h} d^{D-1}x \left(\frac{1}{72}\breve{R}^2 - \frac{1}{180}\breve{R}_{\mu\nu}\breve{R}^{\mu\nu} + \frac{1}{180}\breve{R}_{\mu\nu\lambda\rho}\breve{R}^{\mu\nu\lambda\rho} - \frac{1}{6}\breve{R}V + \frac{1}{2}V^2\right) \tag{3.61}$$

The relation (3.53) for $n = 2$ is

$$\breve{A}_2 = \beta \left(\breve{a}_{0,2} + \frac{1}{2}\breve{a}_{2,3} + \frac{3}{4}\breve{a}_{4,4}\right). \tag{3.62}$$

The coefficients $\breve{a}_{0,2}$, $\breve{a}_{2,3}$, $\breve{a}_{4,4}$ can be found by using results of Refs. [50], [51], i.e.

$$\breve{a}_{0,2} = \int \sqrt{h} d^{D-1}x \left(\frac{1}{72}\bar{R}^2 - \frac{1}{180}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{1}{180}\bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} - \frac{1}{6}\bar{R}V + \frac{1}{2}V^2\right), \tag{3.63}$$

$$\begin{aligned}\breve{a}_{2,3} = \int \sqrt{h} d^{D-1}x \Big(&\frac{1}{12}F^{\mu\nu}F_{\mu\nu}V + \frac{1}{60}F_{\mu\rho}{}^{||\rho}F^{\mu\lambda}{}_{||\lambda} - \frac{1}{72}(F^{\mu\nu}F_{\mu\nu})\bar{R} \\ &-\frac{1}{180}\bar{R}^{\lambda\nu\mu\rho}F_{\lambda\nu}F_{\mu\rho} - \frac{1}{90}\bar{R}_{\mu\nu}F^{\rho\mu}F_{\rho}{}^{\nu}\Big),\end{aligned} \tag{3.64}$$

$$\breve{a}_{4,4} = \int \sqrt{h} d^{D-1}x \left(\frac{1}{288}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{1}{360}F^{\lambda\nu}F^{\mu\rho}F_{\lambda\mu}F_{\nu\rho}\right). \tag{3.65}$$

Here the symbol $||$ corresponds to covariant differentiation with respect to the metric $h_{\mu\nu}$. By using (3.62)–(3.65) one can check that (3.61) holds [48].

Formulas (3.53) are also interesting in applications to genuine Kaluza–Klein theories. In this case the Euclidean space $\mathcal{M}_E$ has to be identified with a higher dimensional space, the Euclidean time with an extra coordinate and $\beta$ with a compactification radius.

# 4 Generalization and external gauge fields

As we have seen the Euclidean formulation of statistical mechanics offers an alternative derivation of relations (2.56), (2.60) between physical and fiducial spectral densities. The advantage of this derivation is that actually it does not depend on the form of the Euclidean wave operator $L_E$. Hence, one can conjecture that (2.60) is more universal and may be true (with some additional assumptions) when $L_E$ is a higher-order elliptic operator. If this were the case such results of Section 2 as spectral asymptotics could be extended to non-linear spectral problems of a generic form. We will not analyze this possibility here but mention another problem where (2.56) and (2.60) are satisfied. This problem is interesting for physical applications.

Consider a charged scalar field in Minkowski space-time which interacts with a static electric potential $\varphi(x^i)$. The equation of motion is

$$\left(-\mathcal{D}_\mu\mathcal{D}^\mu + m^2\right)\phi = 0, \tag{4.1}$$

where $\mathcal{D}_\mu = \partial_\mu - igA_\mu$, $A_\mu dx^\mu = \varphi dt$ and $g$ is the charge of the field. For single-particle excitations $\phi_\omega(t,x^i) = e^{-it\omega}\phi_\omega(x^i)$ is reduced to the non-linear spectral problem

$$\left(\omega^2 - H^2(\omega)\right)\phi_\omega(x^i) = 0, \tag{4.2}$$

$$H^2(\omega) = -\partial_i^2 - g^2\varphi^2 - 2\omega g\varphi + m^2. \tag{4.3}$$

There must be some physical restrictions on the potential $\varphi$ so as to avoid complex energies $\omega$ in (4.2). Complex $\omega$ correspond to an instability of the system (creation of particle–anti-particle pairs, for example). To exclude such effects we assume that electric field is sufficiently weak and vanishes at spatial infinity. Note that equation (4.1) preserves its form under the gauge transformations $\phi' = e^{-igat}\phi$, $\varphi' = \varphi + a$, where $a$ is a constant. These transformations shift the spectrum of $\omega$ in (4.2) to $\omega + ga$. We will eliminate this arbitrariness by requiring that $\varphi$ is zero at infinity.

One can now proceed as before and formulate a corresponding fiducial problem

$$\left(-\partial_\mu\partial^\mu - g^2\varphi^2 - 2g\lambda\varphi + m^2\right)\phi^{(\lambda)} = 0. \tag{4.4}$$

The Klein–Gordon inner product for physical and fiducial fields are determined, respectively, by the currents

$$j_\mu(\phi_1,\phi_2) = -i(\phi_1^*\mathcal{D}_\mu\phi_2 - (\mathcal{D}_\mu\phi_1)^*\phi_2), \tag{4.5}$$

$$\tilde{j}_\mu(\phi_1,\phi_2) = -i(\phi_1^*\partial_\mu\phi_2 - \partial_\mu\phi_1^*\phi_2). \tag{4.6}$$

It should be noted that for charged fields one has to consider two Hamiltonians: one is $H(\omega)$ defined in (4.3) and the other is $H(-\omega)$, for $\omega > 0$. The reason is that the system contains particles and antiparticles which have different charges and interact with the electric field in different ways. The particles and antiparticles are described by $H(\omega)$ and $H(-\omega)$, respectively. The total spectral density $\Phi(\omega)$ is a sum of the spectral densities of particles, $\Phi_+(\omega)$, and antiparticles $\Phi_-(\omega)$. The fiducial problem for antiparticles has to be formulated as (4.4) with $\lambda$ replaced by $-\lambda$. Suppose that $\Phi_\pm(\omega;\lambda)$ are fiducial spectral densities for particles and antiparticles and $\Psi_\pm(\omega;\lambda)$ are corresponding auxiliary densities defined by (2.60). Then by using (4.5), (4.6) one can prove relation (2.56), i.e. that $\Phi_\pm(\omega) = \Psi_\pm(\omega;\omega)$. The asymptotic expansion for the total density $\Phi_+(\omega) + \Phi_-(\omega)$ is given by (2.66), (2.67). The spectral coefficients are $c_n = c_{+,n} + c_{-,n}$ where $c_{\pm,n}$ are determined by the heat kernel asymptotic expansion of operators $H^2(\lambda)$ and $H^2(-\lambda)$. These results can be used to show that an electric field yields a correction $T^2\varphi^2$ to the high-temperature asymptotics (2.84).

One can come to the same result by considering Euclidean theory. Contributions from particles and antiparticles in this case appear in Euclidean $\zeta$-function, (3.16), as integrations along contours $C_+$ and $C_-$ lying in the upper and lower parts of the complex plane, respectively. For charged fields these integrations do not coincide.

# 5 Quantum fields near black holes

## 5.1 Motivations

As we have already discussed, quantum fields in thermal equilibrium with a black hole near its horizon appear, to a static observer, as a system at high temperatures. The free energy of such fields is described by high-temperature asymptotic in a form like (2.84). The local temperature $T$ becomes infinite at the horizon. To avoid this divergence the integration in (2.84) has to be stopped at some small (proper) distance $l$ near the horizon. It is easy to estimate that the entropy of the gas is of order of $\mathcal{A}/l^2$ where $\mathcal{A}$ is the surface area of the horizon. As was first pointed out by 't Hooft [52], if $l$ is associated with the Planck length $l_{Pl}$ the entropy of the gas is of the same order as the entropy of a black hole $S^{BH} = \mathcal{A}/4G$ (here $G = l_{Pl}^2$ is the Newton constant).

The entropy of black holes $S^{BH}$ was introduced by Bekenstein [53] and Hawking [54] who used the fact that black holes possess properties similar to the properties of thermodynamical systems [55]. In a classical theory, however, a Schwarzschild black hole is nothing but an empty space with a strong gravitational field. It still remains a fundamental problem how to identify microscopical degrees of freedom of a black hole responsible for $S^{BH}$. The observation by 't Hooft is important because it offers one of the possible explanations of the Bekenstein-Hawking entropy.

We do not give here a detailed analysis of this idea and do not include a (still growing) list of related references. This can be found in a review paper [56]. In remaining sections we discuss a single problem, i.e., the correspondence between canonical, $F^C$, and Euclidean, $F^E$, free energies in external regions of black holes. In particular we discuss the duality property between infrared thermal divergencies of $F^C$ and ultraviolet divergencies of $F^E$ caused by conical singularities. This is an interesting feature which is important in finding connection between $S^{BH}$ and the entropy of a thermal atmosphere around a black hole.

Note that more detailed analysis of this problem is presented in Refs. [11],[56]. Our aim here is to sketch the main results and pay attention to features related to rotation of a black hole.

## 5.2 Killing horizons

We begin with definitions. Consider a space-time $\mathcal{M}$ which possesses a one-parameter group of isometries parametrized by $t$ and generated by a Killing vector field $\xi^\mu$. Suppose the isometries leave fixed each point of a smooth space-like two-surface $\Sigma$. The surface $\Sigma$ is called a fixed point set of $\xi^\mu$. It can be shown [57] that existence of $\Sigma$ implies the existence of a Killing horizon on $\mathcal{M}$. The Killing horizon consists of two hypersurfaces spanned by null geodesics and the Killing field is tangent to these geodesics. The two hypersurfaces orthogonally intersect at $\Sigma$, and that is why $\Sigma$ is a bifurcation surface for orbits generated by $\xi$. The structure of these orbits near $\Sigma$ is similar to orbits of Lorentz boosts in Minkowski space-time.

The event horizon of black hole solutions in Einstein theory is the Killing horizon[21]. The Killing horizon divides the space-time into four regions: two regions $\mathcal{F}$ and $\mathcal{P}$ where $\xi$ is space-like, and two causally independent regions $\mathcal{L}$ and $\mathcal{R}$ where $\xi$ is time-like. In case of black hole space-times one of such regions, say $\mathcal{R}$, describes the exterior region of a black hole. In such a region near $\Sigma$ the metric can be written in the form

$$ds^2 = -B(dt + a_\alpha dx^\alpha)^2 + d\rho^2 + \sigma_{\alpha\beta}dx^\alpha dx^\beta, \tag{5.1}$$

where $x^\alpha$ is a set of $D-2$ coordinates and $\Sigma$ is the surface $\rho = 0$. The coordinate $\rho$ is the distance measured along a geodesic normal to $\Sigma$ between a point with coordinates $\rho, x^\alpha$ and a point on $\Sigma$ with coordinates $x^\alpha$. The components of (5.1) do not depend on $t$. One can show that $B \simeq \kappa^2\rho^2$ at small $\rho$. Here $\kappa$ is the surface gravity, a constant which can be also defined in a covariant way in terms of the Killing field

$$\kappa = \left[-\frac{1}{2}\xi_{\mu;\nu}\xi^{\mu;\nu}\right]_{\xi^2=0}. \tag{5.2}$$

The Euclidean space $\mathcal{M}^E$ corresponding to $\mathcal{M}$ is obtained by the Wick rotation as we described earlier. The analog of $\Sigma$ on $\mathcal{M}^E$ is a hypersurface which is a fixed-point set of the Euclidean Killing vector field $\xi^E$. The Euclidean metric can be written in a form similar to (5.1) where $t$ is replaced by $-i\tau$. One can easily see that there are conical

[21] Note, however, that we do not require in this Section that the metric obeys the Einstein equations.

singularities on $\Sigma$ if the period of Euclidean time $\tau$ is arbitrary. $\mathcal{M}^E$ is regular if the period is $2\pi/\kappa$.

The geometry near Lorentzian or Euclidean horizons can be characterized by geometric invariants obtained by projecting components of the Riemann tensor and its derivatives on directions orthogonal to $\Sigma$. The corresponding projector is defined in terms of two vectors $l$ and $p$ normal to $\Sigma$ as $P^{\mu\nu} = l^\mu l^\nu \mp p_\mu p_\nu$, where $l^2 = 1$, $p^2 = \mp 1$, $(l \cdot p) = 0$. Signes $-$ and $+$ correspond to Lorentzian and Euclidean theoriues respectively.

## 5.3 Canonical free energy

In the presence of a Killing horizon, the spectrum of single-particle energies $\omega$ has a number of specific features. To better understand them let us first return to the high-temperature asymptotic (2.84). The only difference between stationary and static space-times in this formula is the presence of the term $T^2\Omega^2$ which appears when the Killing frame rotates with the local angular velocity $\Omega$. By using the definition of $\Omega$ (see (2.12)) and the asymptotic form $B \sim \kappa^2\rho^2$ in (5.1) it can be easily shown that $\Omega^2 \sim \rho^2$ near the horizon and, hence, the term $T^2\Omega^2$ is finite at $\rho = 0$. This means that the contribution of rotation in (2.84) can be neglected as compared to other terms (proportional to $T^4$ and $T^2$). Hence, fiducial potential $a_\alpha$ in (2.81) can be ignored and stationary space-time $\mathcal{M}$, Eqs. (2.6)–(2.9), can be replaced by a fiducial static space-time $\tilde{\mathcal{M}}$, Eq. (2.23). Really, *near the horizon one has effectively a static problem* [58].

The last statement can be also formulated in geometrical terms. Consider three invariants $P^{\mu\nu}P^{\lambda\rho}R_{\mu\lambda\nu\rho}$, $P^{\mu\nu}R_{\mu\nu}$, and $R$ computed at $\Sigma$, where $R_{\mu\nu\lambda\rho}$, $R_{\mu\nu}$ and $R$ are $D$–dimensional Riemann, Ricci and scalar curvatures, respectively. It can be shown [58] that these quantities computed on $\mathcal{M}$ coincide with corresponding quantities computed on $\tilde{\mathcal{M}}$. Let us also note that surface gravities $\kappa$ of the Killing horizons as well as the bifurcation surfaces on $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are identical.

To understand what happens near the horizon let us neglect for a moment the curvatures near $\Sigma$ and approximate the metric by the metric on the Rindler space

$$ds^2 = -\kappa^2\rho^2 dt^2 + d\rho^2 + dz_1^2 + ... + dz_{D-2}^2. \tag{5.3}$$

As was explained in section 2.2. the single-particle Hamiltonian can be obtained with the help of a conformal transformation which turns (5.3) into an ultrastatic space. From (2.27), (2.30) one finds

$$H^2 = -\nabla_i\nabla^i - \alpha_D^2\kappa^2 + \kappa^2\rho^2 V. \tag{5.4}$$

Here $\nabla_i$ are covariant derivatives on the space $\bar{\mathcal{B}}$ with the metric

$$dl^2 = \kappa^{-2}\rho^{-2}(d\rho^2 + dz_1^2 + ... + dz_{D-2}^2). \tag{5.5}$$

$\bar{\mathcal{B}}$ is a hyperbolic (Lobachevsky) space with constant negative curvature $\bar{R} = -\kappa^2(D-1)(D-2)$. Although to leading approximation the potential $V$ in (5.4) can be neglected,

$H^2$ has an effective tachyonic mass $-\alpha_D^2\kappa^2$ where $\alpha_D = (D-2)/2$. In its turn, $-\nabla^i\nabla_i$ on $\bar{\mathcal{B}}$ has a mass gap which appears due to constant curvature. What happens is that the tachyonic mass exactly cancels the mass gap (see [59],[60] for details). Thus, the spectrum of $H^2$ is continuous and without mass gap.

Because (5.5) is a maximally symmetric space the spectral density of $H^2$ grows as the volume of $\bar{\mathcal{B}}$. The volume divergence is of the infrared type and one can show that it appears in high-temperature asymptotics (2.84) as divergence of the $T^4$ term.

In previous sections our approach to volume divergences was to cut off integrations over the space to have a system in a box. In case of black holes it means that the region of physical space-time whose proper distance $\rho$ from the horizon is smaller than some length $\epsilon$ should not be considered [61]. The drawback of this procedure is that it makes the space-time incomplete. Thus, it makes sense to consider other regularizations which allow one to work on an entire external region of a black hole. The dimensional [11] and Pauli–Villars [62] regularizations are typical examples of this type.

We begin with the dimensional regularization. The idea is that the volume of (5.5) depends on the number of dimensions $D-1$ and it diverges as $\epsilon^{2-D}$ where $\epsilon$ is the cutoff at small $\rho$ ($\rho > \epsilon$). Formally, if $D < 2$, the integral at small $\rho$ converges so one can use $D$ as a regularization parameter. Consider as an example a massive scalar field[22] described by Eq. (2.18) with $V = m^2$. Near the horizon the diagonal element of the trace of the heat kernel of (5.4) behaves as [11]

$$\left[e^{-H^2 t}\right]_{\text{diag}} \sim \frac{e^{-m^2\kappa^2\rho^2 t}}{(4\pi t)^{(D-1)/2}}\left(1 + b_1 t + b_2 t^2 + ...\right), \tag{5.6}$$

$$b_1 \sim \kappa^2\rho^2\left(\frac{1}{6} - \xi\right)R, \quad b_2 = O(\rho^4). \tag{5.7}$$

Other terms in $b_1$ do not result in singularities at $D \to 4$. The integration in the trace near $\rho = 0$ can be written as

$$\int_{\bar{\mathcal{B}}} \sqrt{\bar{g}} d^{D-1}x \sim \frac{1}{\kappa^{D-1}} \int_\Sigma \int_0 \rho^{1-D} d\rho \left[1 + \frac{1}{4}\rho^2 \mathcal{P}\right], \tag{5.8}$$

where $\mathcal{P}$ is the curvature invariant computed on $\Sigma$

$$\mathcal{P} = 2P^{\mu\nu}P^{\lambda\rho}R_{\mu\lambda\nu\rho} - P^{\mu\nu}R_{\mu\nu}. \tag{5.9}$$

In case of stationary but not static space-times $\mathcal{P}$ has to be computed by using the curvatures of the corresponding fiducial static background $\tilde{\mathcal{M}}$. However, as we have mentioned, curvatures $\mathcal{P}$ on $\tilde{\mathcal{M}}$ and $\mathcal{M}$ coincide [58].

Equations (5.6), (5.8) can be used to derive the divergent part of the trace

$$\left[\text{Tr}\ e^{-H^2 t}\right]_{\text{div}} = \frac{\Gamma\left(1 - \frac{D}{2}\right)}{(4\pi)^{(D-1)/2}} \frac{m^{D-4}}{2\kappa t^{3/2}} \int_\Sigma \left[\left(m^2 - \left(\frac{1}{6} - \xi\right)R\right)t - \frac{\mathcal{P}}{4\kappa^2}\right]. \tag{5.10}$$

[22]The mass term eliminates divergencies at large $\rho$.

From (5.10) one can also get the spectral density of $H^2$ in dimensional regularization (see (2.57))

$$\Phi_{\text{div}}(\omega, D) = \frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \frac{m^{D-4}}{\kappa} \int_\Sigma \left[2\left(m^2 - \left(\frac{1}{6}-\xi\right)R\right) - \frac{\omega^2}{\kappa^2}\mathcal{P}\right], \tag{5.11}$$

and from it the free energy

$$F^C_{\text{div}}(\beta, D) = -\frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \frac{\pi^2 m^{D-4}}{3\kappa\beta^2} \int_\Sigma \left[m^2 - \left(\frac{1}{60}\frac{4\pi^2}{\kappa^2\beta^2}\mathcal{P} + \left(\frac{1}{6}-\xi\right)R\right)\right]. \tag{5.12}$$

The divergencies in (5.11), (5.12) appear as poles at $D = 4$. This is a typical property of dimensional regularization.

Expression (5.11) is suitable to go to the Pauli–Villars (PV) regularization. Let us introduce 5 additional auxiliary scalar fields (PV partners): 2 with masses $M_k$ and 3 with masses $M'_r$, and impose two restrictions

$$f(1) = f(2) = 0, \tag{5.13}$$

where

$$f(p) = m^{2p} + \sum_k M_k^{2p} - \sum_r (M'_r)^{2p} = 0. \tag{5.14}$$

Equation (5.13) has the solutions $M_{1,2} = \sqrt{3\mu^2 + m^2}$, $M'_{1,2} = \sqrt{\mu^2 + m^2}$, $M'_3 = \sqrt{4\mu^2 + m^2}$ where $\mu$ is a regularization parameter, see [62]. The regularized density of states in PV method is defined as

$$\Phi(\omega, \mu) \equiv \Phi(\omega, m) + \sum_k \Phi(\omega, M_k) - \sum_r \Phi(\omega, M'_r), \tag{5.15}$$

where $\Phi(\omega, M_k)$ and $\Phi(\omega, M'_r)$ are spectral densities of PV partners. Fields with masses $M'_r$ give negative contributions to the regularized density (as if these scalars had a wrong, Fermi, statistics).

Suppose now that we start with dimensionally regularized theory. At $D \neq 4$ the spectral density for each PV partner is a finite quantity given by (5.11). Although each of these densities has a pole the poles are cancelled in the total density (5.15) by virtue of the constraints (5.13). Hence, one can put in (5.15) $D = 4$ and get in the limit $\mu \gg m$

$$\Phi_{\text{div}}(\omega, \mu) = \frac{1}{(4\pi)^2\kappa} \int_\Sigma \left[2c\mu^2 + \ln\frac{\mu^2}{m^2}\left(\frac{\omega^2}{\kappa^2}\mathcal{P} + 2\left(\frac{1}{6}-\xi\right)R - 2m^2\right)\right], \tag{5.16}$$

where $c = \ln(729/256)$. The corresponding expression for the divergent part of canonical free energy is

$$F^C_{\text{div}}(\beta, \mu) = -\frac{1}{48\kappa\beta^2} \int_\Sigma \left[2c\mu^2 + \ln\frac{\mu^2}{m^2}\left(\frac{1}{60}\frac{4\pi^2}{\kappa^2\beta^2}\mathcal{P} + \left(\frac{1}{6}-\xi\right)R - m^2\right)\right]. \tag{5.17}$$

The Pauli-Villars regularization effectively results in a cutoff of the integrals near the horizon at the proper distance comparable to the inverse mass of PV partners. This fact allows the following interpretation. Near the horizon, where the local temperature becomes greater than $\mu$, PV fields become thermally excited and by virtue of the constraints (5.13) their contribution exactly cancels the contribution of the physical field.

Usually dimensional and Pauli–Villars regularizations are used to regularize the ultraviolet divergencies. As we have seen, in case of black holes they can be also used to avoid infrared singularities near the horizon. We will show now that this is not an accident and the divergences of canonical free energy in these regularizations match precisely ultraviolet divergences of Euclidean free energy.

## 5.4 Heat kernel expansion and conical singularities

Let us denote by $\mathcal{M}_\beta^E$ a Euclidean manifold which possesses a one-parameter group of isometries. Suppose that $\Sigma$ is a fixed-point set of these isometries and that $\mathcal{M}_\beta^E$ has a structure $C_{\kappa\beta} \times R^{D-2}$ near $\Sigma$. Here $C_\alpha$ is a two-dimensional conical space (the value $\alpha = 2\pi$ corresponds to a plane).

One has the following heat kernel expansion of a Laplace operator on $\mathcal{M}_\beta^E$

$$\mathrm{Tr}\; e^{-tL_E} \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^{\infty} (A_n + A_{\beta,n}) t^n. \tag{5.18}$$

Here $A_n$ are standard coefficients defined on the regular domain of $\mathcal{M}_\beta^E$. Additions $A_{\beta,n}$ appear due to conical singularities and accumulate information about local geometry near $\Sigma$. $A_{\beta,n}$ are local functionals on $\Sigma$ expressed in terms of powers of the Riemann tensor and its derivatives projected with the help of $P^{\mu\nu}$. The quantities $\beta A_{\beta,n}$ are polynomial in $\beta^{-1}$. For the operator $L_E = -\nabla^2 + \xi \breve{R}$ one has

$$A_{\beta,0} = 1 \quad , \quad A_{\beta,1} = \frac{\pi}{3\gamma}(\gamma^2 - 1)\breve{\mathcal{A}}, \tag{5.19}$$

$$A_{\beta,2} = \frac{\pi}{3\gamma} \int_\Sigma \left[ \frac{1}{60}(\gamma^4 - 1)\breve{\mathcal{P}} + (\gamma^2 - 1)\left(\frac{1}{6} - \xi\right)\breve{R} \right], \tag{5.20}$$

where $\gamma = 2\pi/(\kappa\beta)$, $\breve{\mathcal{A}}$ is the surface area of Euclidean horizon $\Sigma$ and $\breve{\mathcal{P}}$ is the invariant on $\Sigma$ defined by (5.9). Laplacians on $C_\alpha$ and the coefficient $A_{\beta,1}$ were discussed by Cheeger [63], (see also [33],[31]). The heat kernel expansion on spaces with fixed-point sets was analyzed by Donnely [64]. Heat kernel expansion (5.18) on $\mathcal{M}_\beta^E$ has been studied in Refs. [32], [65], [66] where the explicit form of (5.20) was found. A summary of results for asymptotics for higher spin fields is presented in Ref. [56].

## 5.5 Infrared/ultraviolet duality

Now we define the one-loop effective action $W^E(\beta)$ on $\mathcal{M}^E_\beta$ by using dimensional regularization, i.e.

$$W^E(\beta) = \frac{1}{2}\ln\det(L_E + m^2) = -\int_0^\infty \frac{dt}{t} e^{-tm^2} \mathrm{Tr}\, e^{-tL_E}. \qquad (5.21)$$

For $D \neq 4$ the singular part of (5.21) can be found from (5.18)

$$W^E_{\mathrm{div}}(\beta) = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\frac{e^{-tm^2}}{(4\pi t)^{D/2}}\left(B_0 + tB_1 + t^2 B_2\right), \qquad (5.22)$$

where $B_k = A_k + A_{\beta,k}$. On a regular Euclidean manifold the ultraviolet divergences are determined only by $A_0$, $A_1$, $A_2$ which are proportional to the period $\beta$. So they appear only as divergencies $E_{0,\mathrm{div}}$ of the vacuum part $E_0$ of $F^E(\beta)$, see (1.16). In case of conical singularities the ultraviolet divergences are polynomials in $\beta^{-1}$ because of $A_{\beta,k}$ and they also appear in the thermal part of $F^E(\beta)$. The divergent part $F^E_{\mathrm{div}}(\beta, D)$ can be found from (5.19), (5.20) and (5.22). Note that $F^E_{\mathrm{div}}(\beta, D)$ is a local functional determined by curvatures and geometrical invariants ($\breve{\mathcal{A}}$, $\breve{R}$, $\breve{\mathcal{P}}$) at the Euclidean horizon. For this local functional the Wick rotation described in section 3.1, Eqs. (3.2), has a well-defined meaning and it can be used to go in $F^E_{\mathrm{div}}(\beta, D)$ from Euclidean to Lorenztian theory[23]. As one can further show, after the Wick rotation the thermal part of $F^E_{\mathrm{div}}(\beta, D)$ is identical to the divergent part of the canonical energy (5.12). This coincidence also takes place in the Pauli–Villars regularization, so one can write

$$F^E_{\mathrm{div}}(\beta,\delta) - E_{0,\mathrm{div}}(\delta) = F^C_{\mathrm{div}}(\beta,\delta), \qquad (5.23)$$

where $\delta$ is a regularization parameter ($\delta = D - 4$ for dimensional regularization and $\delta = \mu^{-1}$ for PV regularization).

Relation (5.23) is remarkable. It demonstrates a duality between large-volume infrared divergences which appear in $F^C(\beta)$ and ultraviolet-type divergences which appear in $F^E(\beta)$ as a result of curvature singularities on the Euclidean horizon. This duality is manifest in the Pauli–Villars regularization where the inverse scale of Pauli–Villars masses $\mu^{-1}$ plays the same role as the cutoff parameter of integrals near the horizon.

We conclude our discussion with very brief comments on the problem of statistical explanation of the Bekenstein–Hawking entropy $S^{BH}$ of black holes. Note that the entropy of a field near the black hole horizon can be computed by the standard formula $S = \beta^2\partial_\beta F^C$ at the Hawking temperature $\beta^{-1} = \kappa/(2\pi)$. By using (5.17) one can show that to leading order $S \sim \mathcal{A}/\delta^2$ where $\mathcal{A}$ is the surface area of the black hole horizon and $\delta$ is the cutoff. The infrared/ultraviolet duality enables one to identify $\delta^{-1}$ with a high-energy cutoff and, in particular, with the Planck mass. This fact is interesting

[23] This procedure is analyzed in some detail in [67] for a Kerr–Newman black hole.

because it can be used as one of possible explanations of the Bekenstein–Hawking entropy $S^{BH} = \mathcal{A}/(4G)$. Indeed, suppose that Einstein gravity is entirely induced by quantum effects of some fundamental underlying theory. For simplicity one can assume that the degrees of freedom (constituents) of this theory are some field variables. Then the induced Newton constant is $G \sim \delta^2$ where $\delta^{-1}$ and $S^{BH} \sim \mathcal{A}/\delta^2$. Because of the duality property the same cutoff $\delta$ which determines the induced Newton constant appears also in the entropy of constituents $S$ and automatically $S$ has the same order of magnitude as $S^{BH}$. A similar mechanism may be realized in open string theory [68]. Further discussion of the black hole entropy problem in induced gravity and concrete realizations of this idea can be found in Ref. [69].

# 6 Resume

The aim of these notes was to overview a number of new results and methods in finite-temperature field theory in classical external stationary backgrounds. We have also discussed their physical applications, including quantum effects near rotating black holes. The physics here is reduced to a class of spectral problems which depend on the spectral parameter in a non-linear way. This also represents an interest form mathematical point of view because some results of the spectral theory such as heat kernel asymptotics can be extended to this class of problems. One of our motivations was to find a connection between canonical and Euclidean methods in statistical mechanics in situations more non-trivial than static external fields. It is interesting to note that the Euclidean gravity yields an alternative derivation of the spectral asymptotics. This enables one to avoid a complicated analysis of inner products and measures which has to be done in the Lorenztian theory. This fact hints that our results might be extended to a larger class of spectral problems.

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